Quantum ergodicity in the Benjamini-Schramm limit in higher rank

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joint work with Farrell Brumley, Simon Marshall, and Jasmin Matz

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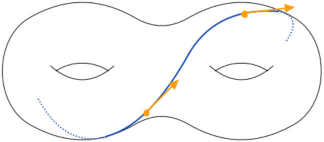
Geodesic flow on hyperbolic surface

- Y compact hyperbolic surface
- $\Phi_t \curvearrowright T^1 Y$ geodesic flow



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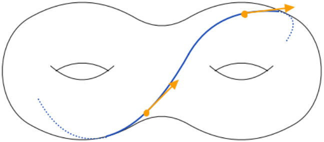

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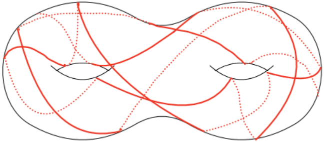

Classical and quantum mechanics on Y

$$\begin{tikzcd}
\text{classical mechanics} \approx \underset{\text{geodesic flow}}{\Phi_t \curvearrowright T^1 Y} \arrow[d, bend right, "\text{quantization}"'] \\
\text{quantum mechanics} \approx \underset{\text{Schrödinger flow}}{e^{ith\Delta} \curvearrowright L^2(Y)} \arrow[u, bend right, dashed, "h \to 0"']
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Quantum particles

- Renormalize volume measure: $đ\mathrm{Vol} = \frac{d\mathrm{Vol}}{\mathrm{Vol}(Y)}$
- Quantum particle $\rightsquigarrow \psi \in L^2(Y, đ\mathrm{Vol})$ with $||\psi||_2 = 1$

$$\begin{aligned} \mathbb{P}(\text{observing } \psi \text{ in } E \subset Y) &= \int_E |\psi|^2 \, đ\mathrm{Vol} \\ &= \int_Y \mathbb{1}_E \cdot |\psi|^2 \, đ\mathrm{Vol} \end{aligned}$$

- If ψ were *equidistributed*:

$$\begin{aligned} \mathbb{P}(\text{observing } \psi \text{ in } E \subset Y) &= \frac{\mathrm{Vol}(E)}{\mathrm{Vol}(Y)} \\ &= \int_Y \mathbb{1}_E \, đ\mathrm{Vol} \end{aligned}$$

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The Laplacian

- Eigendata of Δ:

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \qquad \text{eigenvalues of } \Delta$$

$$\{\psi_j\} \qquad \text{ONB of eigenfunctions of } \Delta$$

- In QM, ψ_j has energy $h^2\lambda_j$. Let $h_j = \frac{1}{\sqrt{\lambda_j}}$.

$$\text{fix } h \text{ and let } \lambda_j \to \infty \quad \approx \quad \text{fix energy and let } h_j \to 0$$

- As $\lambda_j \to \infty$, should "recover" ergodicity $\rightsquigarrow$ ψ_j *equidistributes*

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Theorem (QE in the large eigenvalue limit; Snirelman, Zelditch, Colin de Verdiere)

Let $a \in C^\infty(Y)$. Then

$$\lim_{\lambda\to\infty} \frac{1}{\#\{j : \lambda_j \leq \lambda\}} \sum_{j:\lambda_j\leq\lambda} \left| \int_Y a \cdot |\psi_j|^2 \,\text{đVol} - \int_Y a \,\text{đVol} \right|^2 = 0.$$

- Average over eigenfunctions with eigenvalue less than λ
- Compare the measures $|\psi_j|^2$đVol and đVol weakly (integrate against test function)
- Interpretations:
  1. Generic high energy quantum particles equidistribute.
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Visualization of quantum ergodicity

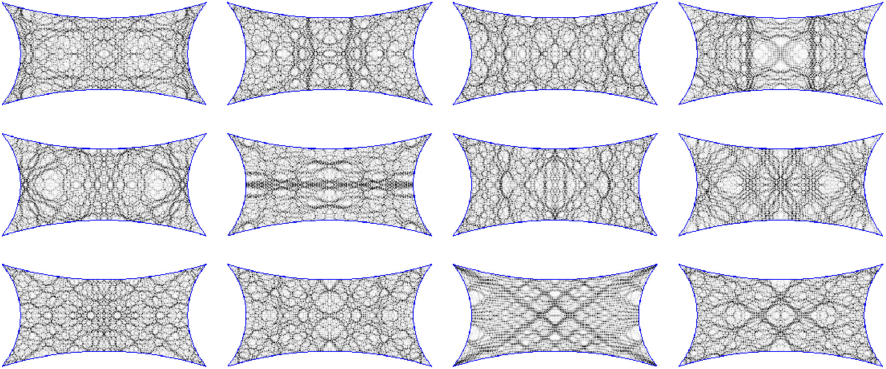

Figure: Image made by Alex Barnett

QE in large eigenvalue limit vs. QE in the BS limit

- Eigenvalues of Δ lie in $[0, \infty)$
- QE in the large eigenvalue limit:

fix the manifold & vary the spectral window

- QE in the Benjamini-Schramm limit:

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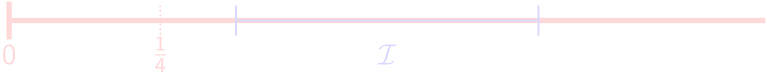

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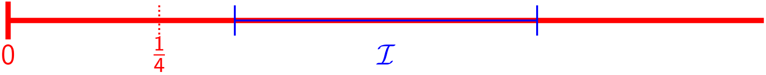
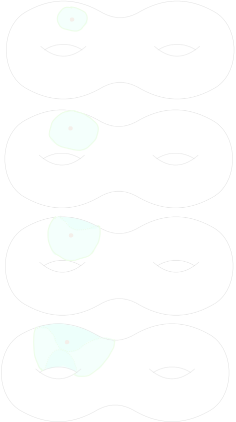


Benjamini-Schramm convergence

(Y_n) *Benjamini-Schramm converges* to $\mathbb{H}$ if, for every $R > 0$,

$$\lim_{n\to\infty} \frac{\mathrm{Vol}\big(\{y \in Y_n : \mathrm{InjRad}_{Y_n}(y) \leq R\}\big)}{\mathrm{Vol}(Y_n)} = 0.$$



Interpretation: most points have arbitrarily large injectivity radius

Spectrum of Δ on $\mathbb{H}$ is $[\frac{1}{4}, \infty)$.

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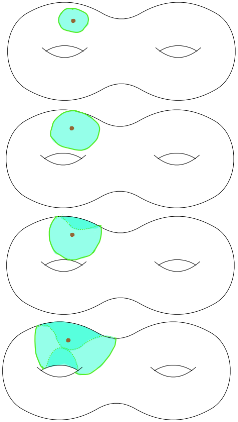

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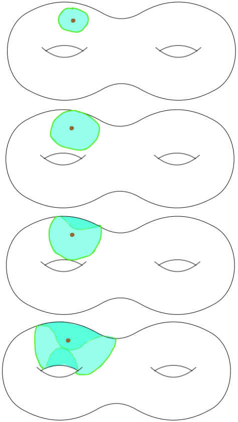

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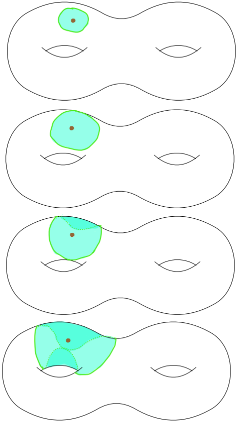

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QE in the BS limit for hyperbolic surfaces

Theorem (Le Masson-Sahlsten '17)

Suppose (Y_n) is a sequence of compact hyperbolic surfaces s.t.

1. *Benjamini-Schramm convergence: $Y_n \xrightarrow{BS} \mathbb{H}$.*
2. *Uniform spectral gap: $\lambda_1^{(n)}$ bounded away from 0 for all n.*
3. *Uniform discreteness: $\mathrm{InjRad}(Y_n)$ bounded away from 0 for all n.*

Let $\{\psi_j^{(n)}\}$ be ONB of eigenfunctions for Δ acting on $L^2(Y_n)$ with eigenvalues $0 = \lambda_0^{(n)} \le \lambda_1^{(n)} \le \dots$. Let $\mathcal{I} \subset (\frac{1}{4}, \infty)$ be a compact subinterval. Let $a_n \in L^\infty(Y_n)$ with uniformly bounded L^∞-norm. Then

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	rank one	higher rank
archimedean	hyperbolic surfaces	symmetric spaces
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Quantization in higher rank

rank one → geod. flow ergodic ($\mathbb{R}$-action) → QE involves Δ

higher rank → geod. flow NOT ergodic (BUT ergodic $\mathbb{R}^k$-action) → QE involves k operators

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Symmetric spaces

- G = semisimple Lie group over $\mathbb{R}$ (w/o compact factors)
- X is associated Riemannian manifold called *symmetric space*
- $X = G/K$ with K a maximal compact subgroup
- $D(G, K)$ = G-invariant differential operators on X
- Fact: $D(G, K)$ generated by k operators

$G = \mathrm{SL}(2, \mathbb{R})$

$X = \mathbb{H}$

$K = \mathrm{SO}(2)$

$D(G, K)$ = algebra generated by Δ

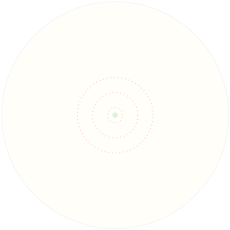

Figure: Δ closely related to averaging over spheres in $\mathbb{H}$

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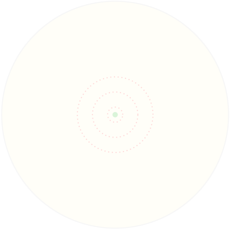

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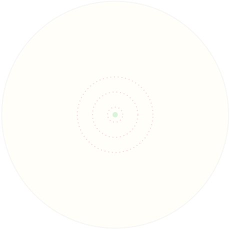

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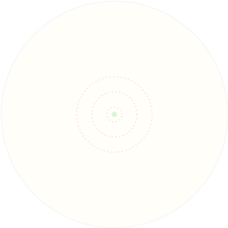

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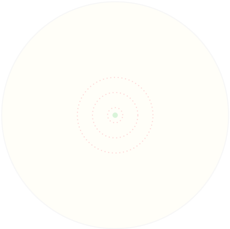

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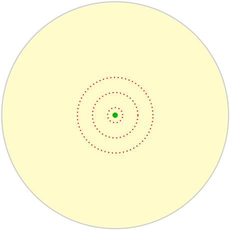

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- G = semisimple algebraic group over F (non-archimedean local field)
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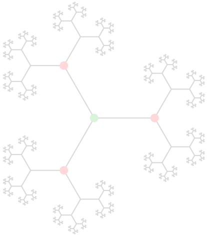

Figure: Adjacency operator $\mathcal{A}$ on tree involves summing over sphere of radius 1

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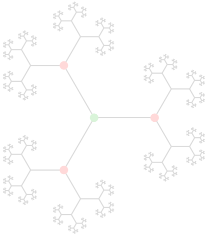

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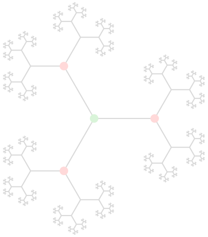

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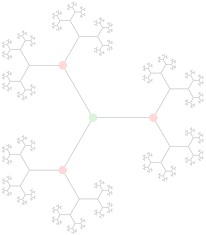

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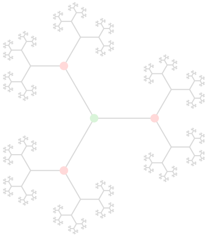

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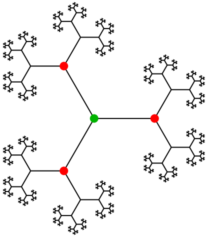

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Buildings are composed of branching apartments

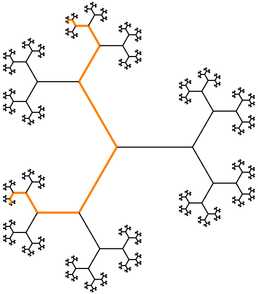
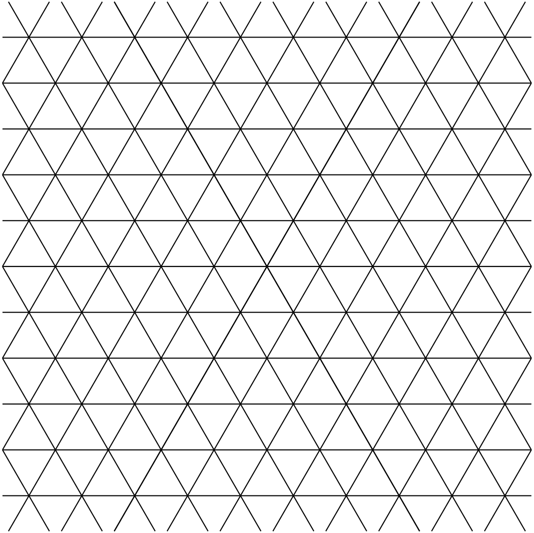


Figure: An apartment in the tree is a bi-infinite geodesic.

An apartment in the Bruhat-Tits building of SL(3)



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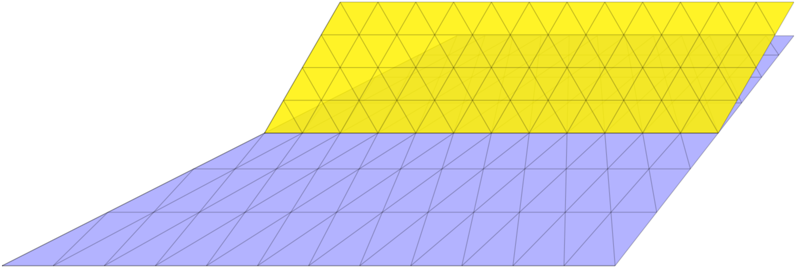


Visualization of Bruhat-Tits building for SL(3)

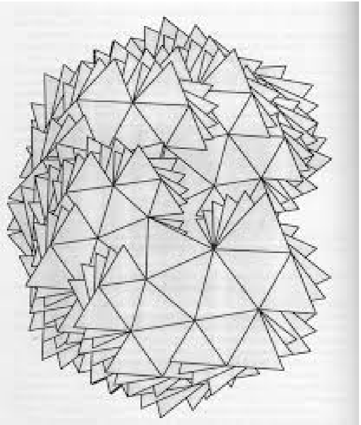
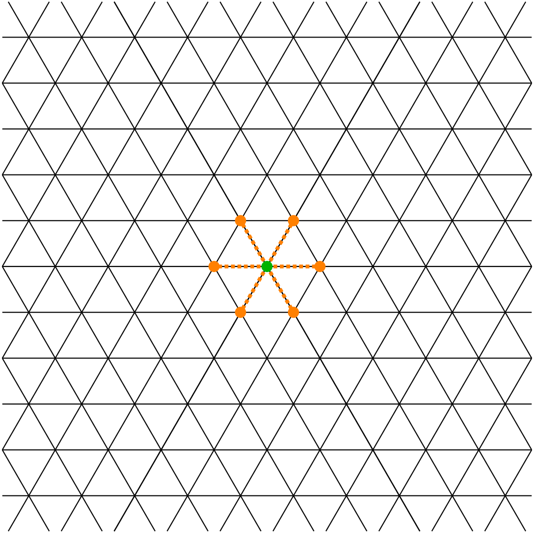


Figure: Image made by Paul Garrett

$H(G, K)$ generated by refinements of adjacency operator



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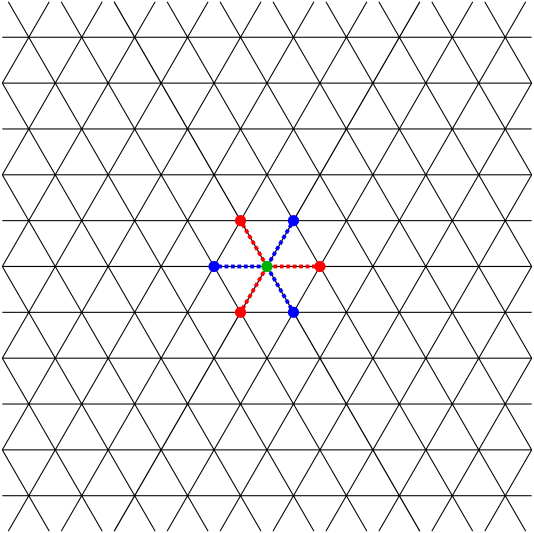
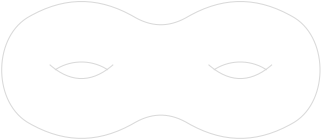
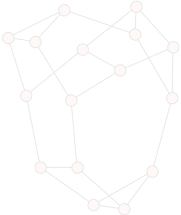


Quotients of X and $\mathcal{B}$

- $\Gamma < G$ cocompact, torsionfree lattice

$$\Gamma \backslash G / K \text{ is } \begin{cases} \text{locally symmetric space (e.g. hyperbolic surface)} \\ \text{finite simplicial complex (e.g. finite regular graph)} \end{cases}$$





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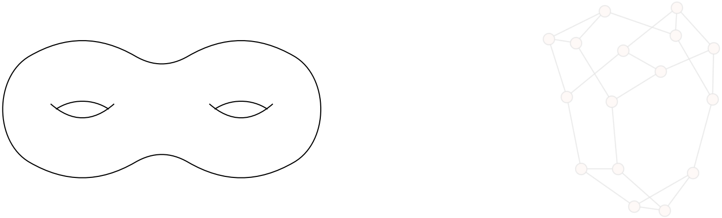

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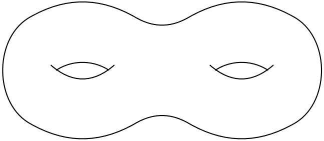

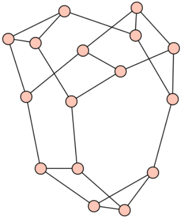

Joint eigenfunctions and spectral parameters

- Let $\mathcal{C}$ = either $D(G, K)$ or $H(G, K)$
- $\mathcal{C}$ generated by k operators $A_1, \ldots, A_k$

$$\mathcal{C} \curvearrowright L^2(\Gamma \backslash G/K) = \bigoplus_j \mathbb{C}\psi_j \quad \text{(joint eigenfunctions)}$$

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- Tempered spectrum:

$$\Omega^{+}_{\text{temp}} \leftrightarrow \left\{\begin{matrix}\text{spectrum of} \\ \mathcal{C} \curvearrowright L^2(G/K) \\ \text{universal cover}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{spherical repns} \\ \text{in } L^2(G)\end{matrix}\right\}$$

Figure: Ω^{+}_{temp} for Δ on $\mathbb{H}$ is $[1/4, \infty)$

Figure: Ω^{+}_{temp} for $\mathcal{A}$ on $(q+1)$-regular tree is $[-2\sqrt{q}, 2\sqrt{q}]$

Tempered spectrum

$$\{k\text{-tuples}\} \leftrightarrow \left\{\begin{matrix}\text{irreps of } \mathcal{C} \\ \text{(all 1-dim.)}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{admissible irreps of } G \\ \text{with } K\text{-fixed vector} \\ (\textit{spherical repns})\end{matrix}\right\}$$

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BS convergence implies Plancherel convergence

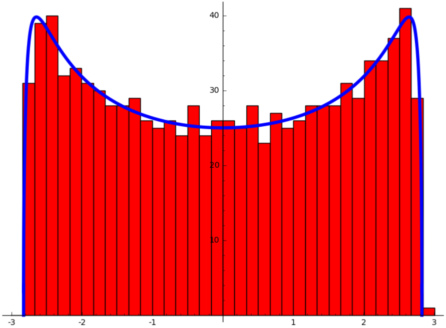
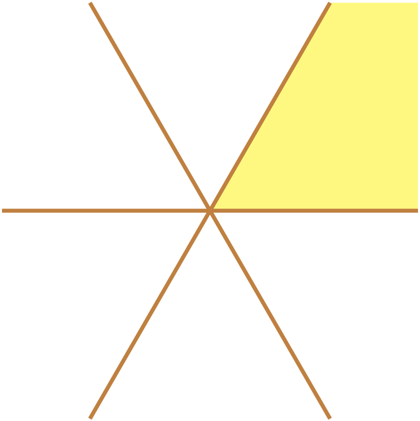


Figure: Distribution of eigenvalues for large random 3-regular graph

$$\frac{\#\{j : \lambda_j^{(n)} \in \mathcal{I}\}}{\mathrm{Vol}(Y_n)} \to \mu(\mathcal{I})$$

Tempered spectrum for symmetric spaces

- For symmetric spaces, the tempered spectrum is parametrized by $\mathfrak{a}^*/W$, i.e. a Weyl chamber.



Framework for QE in the BS limit

Suppose $Y_n = \Gamma_n \backslash \mathbb{H}$ with Γ_n cocompact, torsionfree lattices s.t.

1. Benjamini-Schramm convergence: $Y_n \xrightarrow{BS} \mathbb{H}$
2. Uniform spectral gap for $\Delta \curvearrowright L^2(Y_n)$
3. Uniform discreteness

For each Y_n let $\{\psi_j^{(n)}\}$ be ONB of eigenfunctions of $\Delta \curvearrowright L^2(Y_n)$ with associated eigenvalues $\lambda_j^{(n)}$. Let $\mathcal{I} \subset (1/4, \infty)$ be a compact interval. Let $a_n \in L^\infty(Y_n)$ with uniform L^∞-bound. Then we expect

$$\lim_{n\to\infty} \frac{1}{\#\{j : \lambda_j^{(n)} \in \mathcal{I}\}} \sum_{j:\lambda_j^{(n)} \in \mathcal{I}} \left| \int_{Y_n} a_n \cdot |\psi_j^{(n)}|^2 \, đ\mathrm{Vol} - \int_{Y_n} a_n \, đ\mathrm{Vol} \right|^2 = 0.$$

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Main Theorem (Brumley-Marshall-Matz-P. '25)

Suppose G is a product of non-compact, connected, centerless, simple real Lie groups. Let $X = G/K$ be the symmetric space. Let $\Gamma_n < G$ be a sequence of irreducible, cocompact, uniformly discrete, torsion free lattices. Let $Y_n = \Gamma_n \backslash X$. Let G_1 be a simple factor of G. Assume that

1. Benjamini-Schramm convergence: $Y_n \to G/K$.
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Let $\psi_j^{(n)}$ be an ONB of joint eigenfunctions of $D(G, K)$ acting on $L^2(Y_n)$ with spectral parameters $\nu_j^{(n)}$. There exists a finite W-invariant set of subspaces $\{P_i\}$ of $\mathfrak{a}^*$ $(= \Omega^+_{\text{temp}})$ such that for any compact $\Theta \subset \mathfrak{a}^* \setminus \bigcup_i P_i$ with non-empty interior and any norm-bounded sequence of $a_n \in L^\infty(Y_n)$,

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- A_M = wave propagator
  - $U_m \approx$ avg over polytopal ball
  - $B_m(x)$ = polytopal ball of radius m centered at x

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Geometric bound in rank one

- In rank one, polytopal balls are just metric balls.
- Let $G/K = \mathbb{T}_{q+1}$. Suppose $d(x, y) = r$. Then

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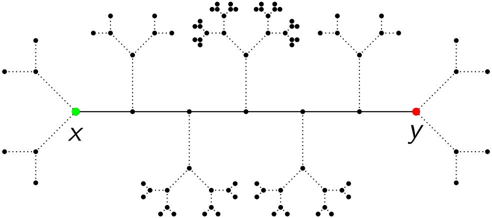

Figure: $B_8(x) \cap B_8(y)$ on 3-regular tree with $d(x,y) = 6$

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- Let $G/K = \mathbb{H}$. Suppose $d(x, y) = r$. Then

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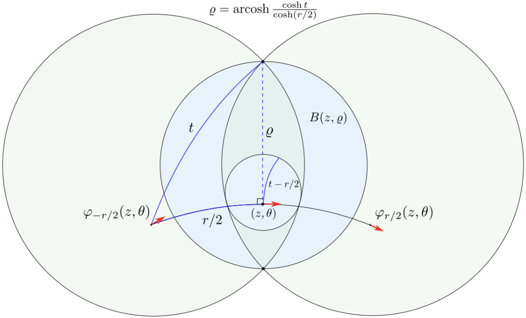

FIGURE 2. The volume of the sets $F_t(r)$ used in the proof of Proposition 4.1 can be controlled by the volume of the balls $B(z, t - r/2)$ and $B(z, \varrho)$, where $\cosh \varrho = \frac{\cosh t}{\cosh(r/2)}$ by the hyperbolic version of Pythagoras' theorem. The volume of both of these balls is $O(e^{t-r/2})$.

Cartan decomposition and relative position

- G = semisimple real Lie group
- Cartan decomposition:

$$G = \bigsqcup_{\lambda \in \mathfrak{a}^+} K e^{\lambda} K.$$

- Weyl chamber-valued "distance":

$$d_{\mathfrak{a}^+}(xK, yK) = \lambda \text{ iff } x^{-1}y \in K e^{\lambda} K$$

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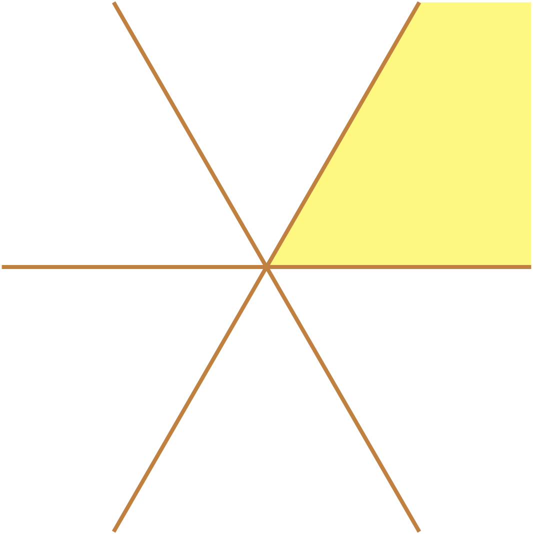

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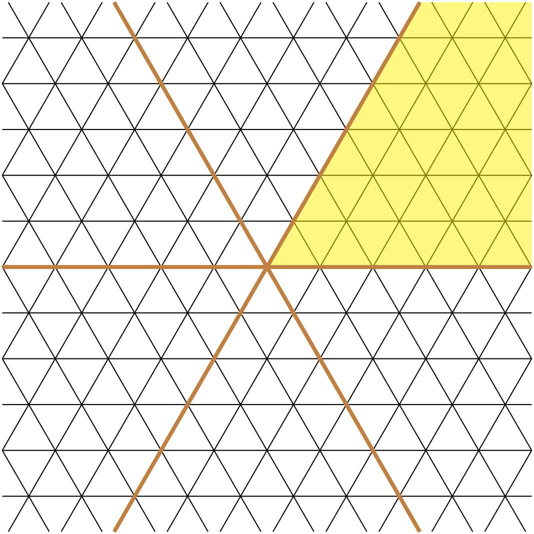

Polytopal balls and directing elements

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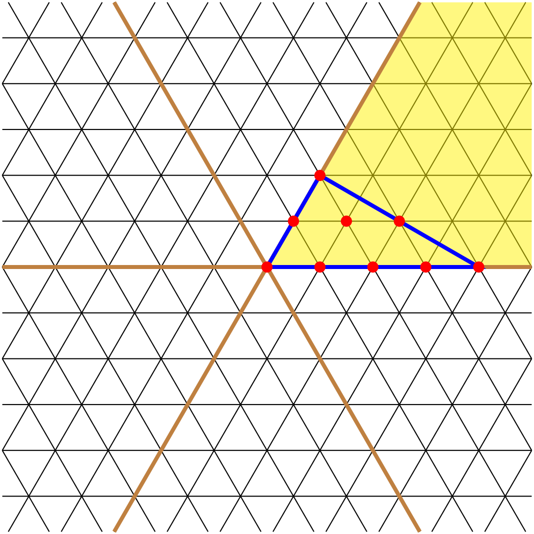
- If P has a unique vertex $H_0 \in \mathfrak{a}^+$ maximizing $\langle \rho, - \rangle$ (half sum of positive roots), then we call H_0 the *directing element*.

Polytopal balls



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- BUT, not true for arbitrary choices of H_0.
- In my thesis, I show that for non-archimedean PGL(3), we have

$$|B_{t\cdot P}(x) \cap B_{t\cdot P}(y)| \approx \Gamma_P(\lambda, t) q^{\langle \rho, 2t\cdot H_0 - \frac{\lambda}{2}\rangle}.$$

- Here $\Gamma_P(\cdot)$ is a piecewise linear function.

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- In the context of the proof technique, one seeks the following bound:

$$\mathrm{Vol}(B_{t\cdot P}(x)\cap B_{t\cdot P}(y))\lesssim e^{\langle\rho,2t\cdot H_0-\lambda\rangle}\quad(\text{or }\lesssim q^{\langle\rho,2t\cdot H_0-\lambda\rangle})$$

- BUT, not true for arbitrary choices of H_0.
- In my thesis, I show that for non-archimedean PGL(3), we have

$$|B_{t\cdot P}(x)\cap B_{t\cdot P}(y)|\approx\Gamma_P(\lambda,t)q^{\langle\rho,2t\cdot H_0-\frac{\lambda}{2}\rangle}.$$

- Here $\Gamma_P(\cdot)$ is a piecewise linear function.
- However, if we choose H_0 to be *extremally singular*, then we get

$$|B_{t\cdot P}(x)\cap B_{t\cdot P}(y)|\lesssim q^{\langle\rho,2t\cdot H_0-\lambda\rangle}.$$

Geometric bound in higher rank

- In joint work with Brumley, Marshall, and Matz, we've been able to show that if one chooses an *extremally singular* directing element H_0, then we get

$$\mathrm{Vol}(B_{t\cdot P}(x) \cap B_{t\cdot P}(y)) \lesssim \log(t)^k e^{\langle \rho, 2t\cdot H_0 - \lambda \rangle}.$$

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- Really, we work with thickened spherical shells with "radius" tH_0, rather than balls.
- The root systems E_6, E_8, F_4, G_2 do not admit extremally singular elements.

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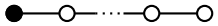
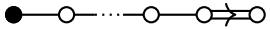
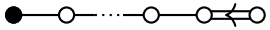
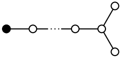
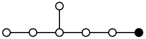
type of Φ	Dynkin diagram of Φ_0 (remove $\bullet$)	type of Φ_0
A_n		A_{n-1}
B_n		B_{n-1}
C_n		C_{n-1}
D_n		D_{n-1}
E_7		E_6