

Quantum ergodicity in the Benjamini-Schramm limit on higher rank real and p -adic locally symmetric spaces

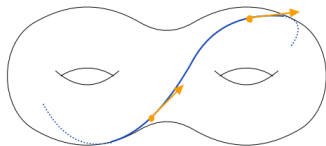
Carsten Peterson

Paderborn University

May 28, 2024

Geodesic flow on hyperbolic surface

- Y compact hyperbolic surface
- $\Phi_t \curvearrowright T^1Y$ geodesic flow

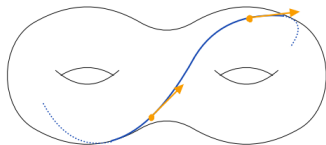


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 \implies generic geodesics *equidistribute*

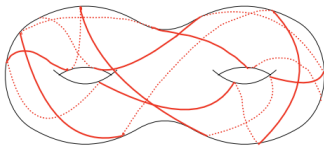


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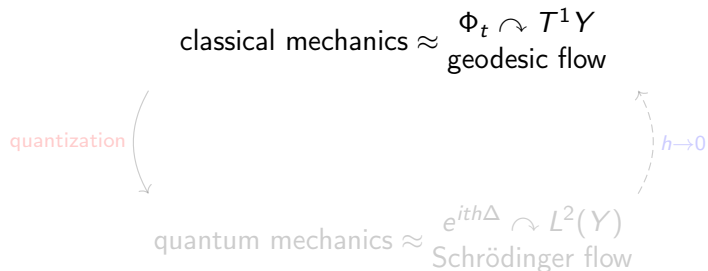
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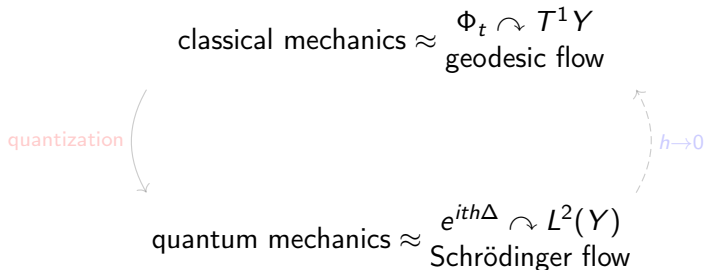
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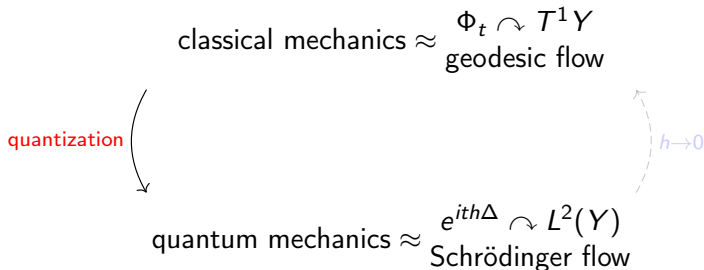
Classical and quantum mechanics on Y



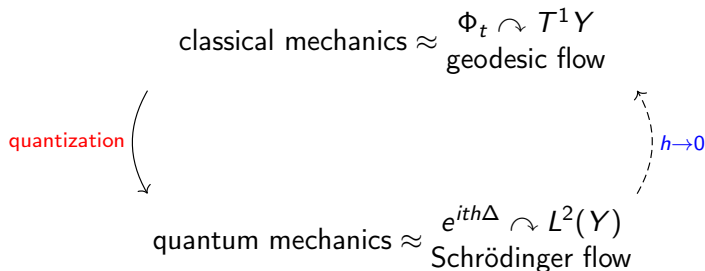
Classical and quantum mechanics on Y



Classical and quantum mechanics on Y



Classical and quantum mechanics on Y



Quantum particles

- Renormalize volume measure: $d\text{Vol} = \frac{d\text{Vol}}{\text{Vol}(Y)}$
- Quantum particle $\rightsquigarrow \psi \in L^2(Y, d\text{Vol})$ with $\|\psi\|_2 = 1$

$$\begin{aligned}\mathbb{P}(\text{observing } \psi \text{ in } E \subset Y) &= \int_E |\psi|^2 d\text{Vol} \\ &= \int_Y \mathbb{1}_E \cdot |\psi|^2 d\text{Vol}\end{aligned}$$

- If ψ were *equidistributed*:

$$\begin{aligned}\mathbb{P}(\text{observing } \psi \text{ in } E \subset Y) &= \frac{\text{Vol}(E)}{\text{Vol}(Y)} \\ &= \int_Y \mathbb{1}_E d\text{Vol}\end{aligned}$$

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The Laplacian

- Eigendata of Δ :

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \quad \text{eigenvalues of } \Delta$$

$$\{\psi_j\} \quad \text{ONB of eigenfunctions of } \Delta$$

- In QM, ψ_j has energy $h^2\lambda_j$. Let $h_j = \frac{1}{\sqrt{\lambda_j}}$.

fix h and let $\lambda_j \rightarrow \infty \approx$ fix energy and let $h_j \rightarrow 0$

- As $\lambda_j \rightarrow \infty$, should “recover” ergodicity $\rightsquigarrow \psi_j$ *equidistributes*

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Quantum ergodicity theorem

Theorem (QE in the large eigenvalue limit; Snirelman, Zelditch, Colin de Verdiere)

Let $a \in C^\infty(Y)$. Then

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\#\{j : \lambda_j \leq \lambda\}} \sum_{j: \lambda_j \leq \lambda} \left| \int_Y a \cdot |\psi_j|^2 \, d\text{Vol} - \int_Y a \, d\text{Vol} \right|^2 = 0.$$

- Average over eigenfunctions with eigenvalue less than λ
- Compare the measures $|\psi_j|^2 d\text{Vol}$ and $d\text{Vol}$ weakly (integrate against test function)
- Interpretations:
 - Generic high energy quantum particles equidistribute.
 - Generic bounded energy quantum particles equidistribute as $h \rightarrow 0$.

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QE in large eigenvalue limit vs. QE in the BS limit

- Eigenvalues of Δ lie in $[0, \infty)$
- QE in the large eigenvalue limit:

fix the manifold & vary the spectral window



- QE in the Benjamini-Schramm limit:

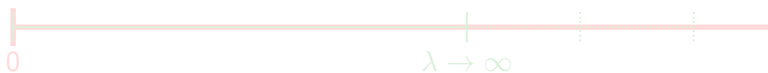
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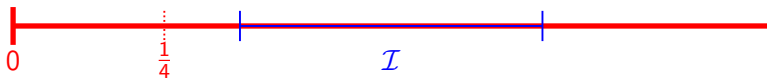
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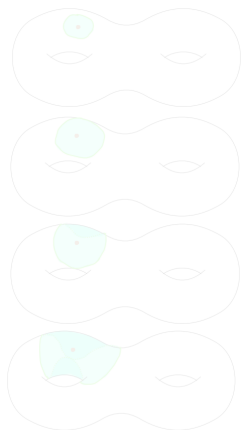
Benjamini-Schramm convergence

(Y_n) Benjamini-Schramm converges to \mathbb{H} if, for every $R > 0$,

$$\lim_{n \rightarrow \infty} \frac{\text{Vol}(\{y \in Y_n : \text{InjRad}_{Y_n}(y) \leq R\})}{\text{Vol}(Y_n)} = 0.$$

Interpretation: most points have arbitrarily large injectivity radius

Spectrum of Δ on \mathbb{H} is $[\frac{1}{4}, \infty)$.



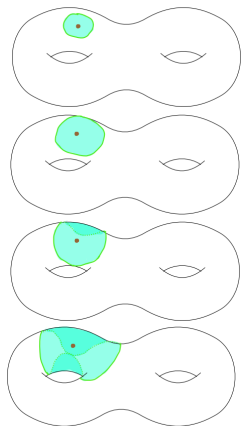
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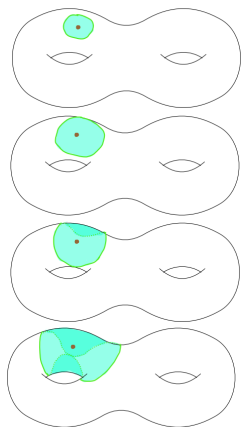
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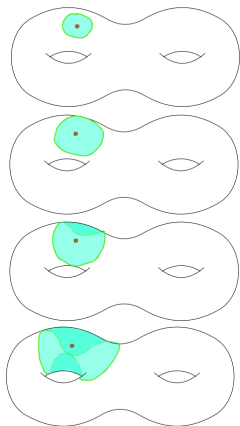
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QE in the BS limit for hyperbolic surfaces

Theorem (Le Masson-Sahlsten '17)

Suppose (Y_n) is a sequence of compact hyperbolic surfaces s.t.

- 1 Benjamini-Schramm convergence: $Y_n \xrightarrow{BS} \mathbb{H}$.
- 2 Uniform spectral gap: $\lambda_1^{(n)}$ bounded away from 0 for all n .
- 3 Uniform discreteness: $\text{InjRad}(Y_n)$ bounded away from 0 for all n .

Let $\{\psi_j^{(n)}\}$ be ONB of eigenfunctions for Δ acting on $L^2(Y_n)$ with eigenvalues $0 = \lambda_0^{(n)} \leq \lambda_1^{(n)} \leq \dots$. Let $\mathcal{I} \subset (\frac{1}{4}, \infty)$ be a compact subinterval. Let $a_n \in L^\infty(Y_n)$ with uniformly bounded L^∞ -norm. Then

$$\lim_{n \rightarrow \infty} \frac{1}{\#\{j : \lambda_j^{(n)} \in \mathcal{I}\}} \sum_{j: \lambda_j^{(n)} \in \mathcal{I}} \left| \int_{Y_n} a_n \cdot |\psi_j^{(n)}|^2 \, d\text{Vol} - \int_{Y_n} a_n \, d\text{Vol} \right|^2 = 0.$$

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	rank one	higher rank
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Quantization in higher rank

rank one \rightarrow geod. flow ergodic
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higher rank \rightarrow geod. flow NOT ergodic
(BUT ergodic \mathbb{R}^k -action) \rightarrow QE involves k operators

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- $G =$ semisimple Lie group over \mathbb{R} (w/o compact factors)
- X is associated Riemannian manifold called *symmetric space*
- $X = G/K$ with K a maximal compact subgroup
- $D(G, K) =$ G -invariant differential operators on X
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$$D(G, K) = \text{algebra generated by } \Delta$$

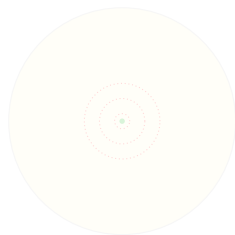


Figure: Δ closely related to averaging over spheres in \mathbb{H}

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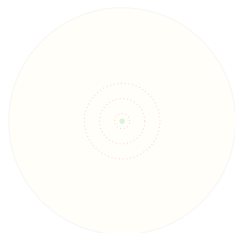


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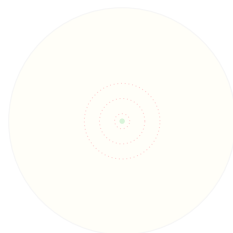


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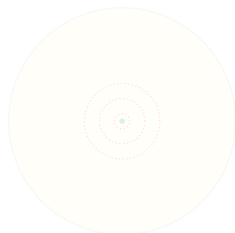


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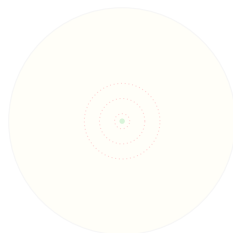


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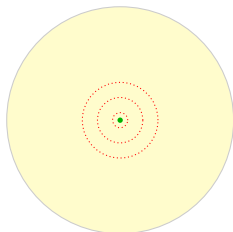


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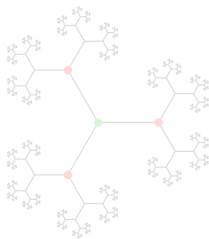


Figure: Adjacency operator \mathcal{A} on tree involves summing over sphere of radius 1

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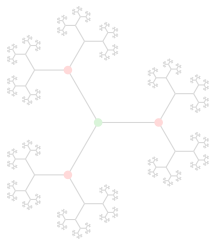


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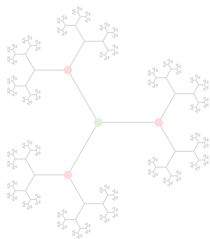


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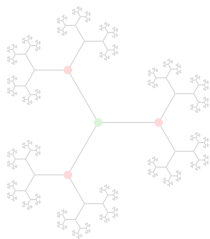


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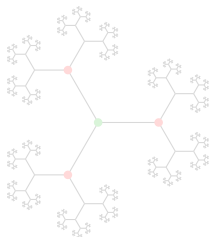


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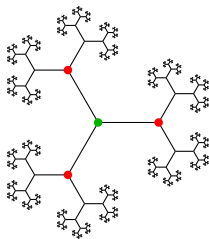


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Buildings are composed of branching apartments

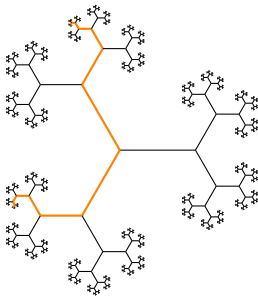
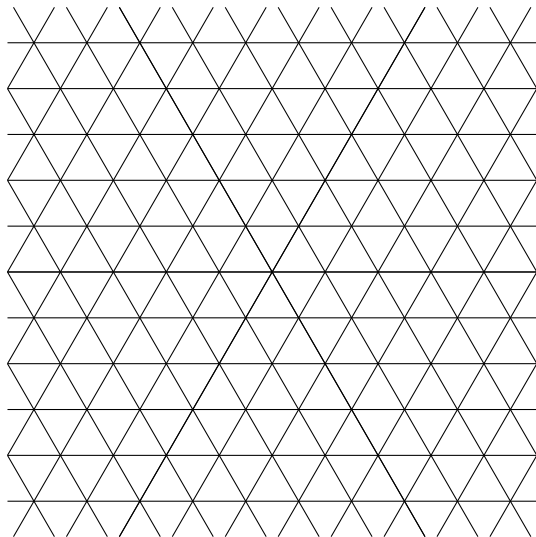


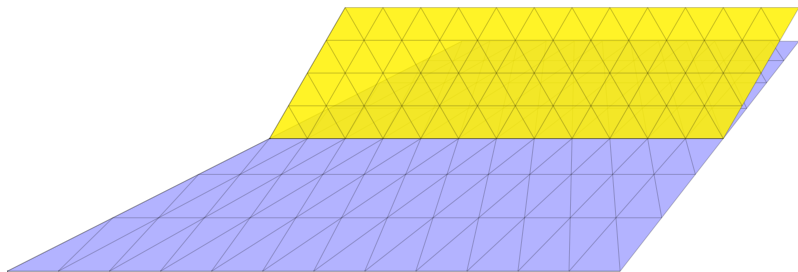
Figure: An apartment in the tree is a bi-infinite geodesic.



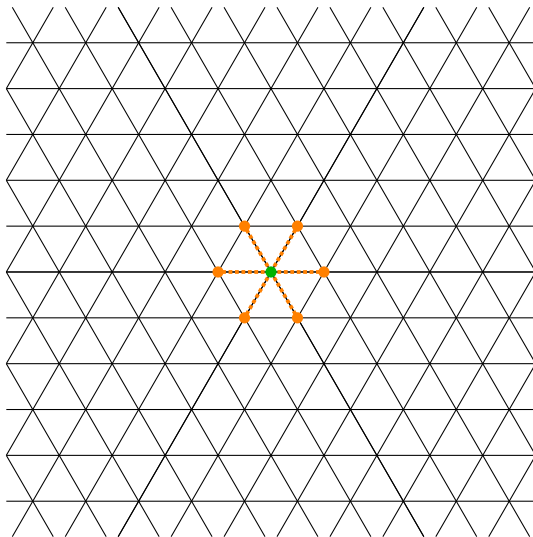
An apartment in the Bruhat-Tits building of $\mathrm{PGL}(3, F)$



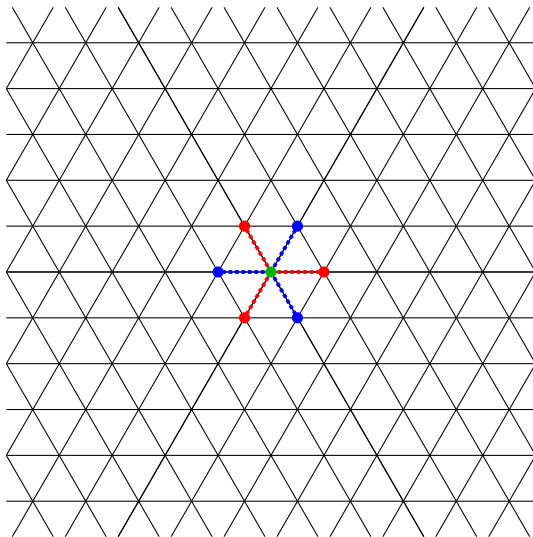
Branching apartments



$H(G, K)$ generated by refinements of adjacency operator



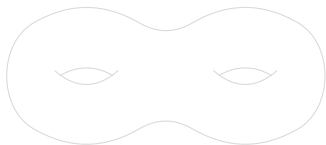
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Quotients of X and \mathcal{B}

- $\Gamma < G$ cocompact, torsionfree lattice

$\Gamma \backslash G/K$ is $\begin{cases} \text{locally symmetric space (e.g. hyperbolic surface)} \\ \text{finite simplicial complex (e.g. finite regular graph)} \end{cases}$

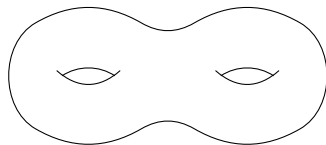


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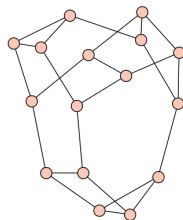
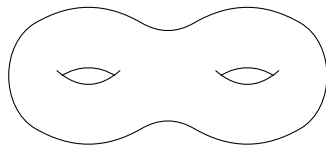


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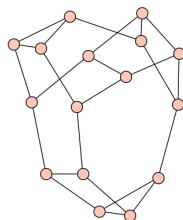
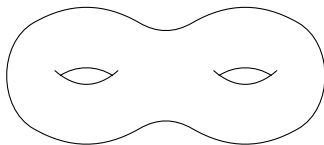


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Joint eigenfunctions and spectral parameters

- Let $\mathcal{C} =$ either $D(G, K)$ or $H(G, K)$
- \mathcal{C} generated by k operators A_1, \dots, A_k

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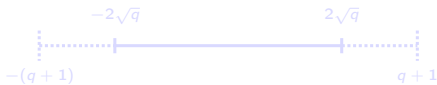


Figure: Ω_{temp}^+ for \mathcal{A} on $(q+1)$ -regular tree is $[-2\sqrt{q}, 2\sqrt{q}]$

Tempered spectrum

$$\{k\text{-tuples}\} \leftrightarrow \left\{ \begin{array}{l} \text{irreps of } \mathcal{C} \\ \text{(all 1-dim.)} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{admissible irreps of } G \\ \text{with } K\text{-fixed vector} \\ \text{(spherical reps)} \end{array} \right\}$$

- Tempered spectrum:

$$\Omega_{\text{temp}}^+ \leftrightarrow \left\{ \begin{array}{l} \text{spectrum of} \\ \mathcal{C} \curvearrowright L^2(G/K) \\ \text{universal cover} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{spherical reps} \\ \text{in } L^2(G) \end{array} \right\}$$



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BS convergence implies Plancherel convergence

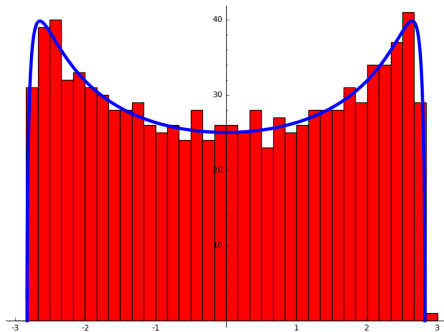


Figure: Distribution of eigenvalues for large random 3-regular graph

$$\frac{\#\{j : \lambda_j^{(n)} \in \mathcal{I}\}}{\text{Vol}(Y_n)} \rightarrow \mu(\mathcal{I})$$

Framework for QE in the BS limit

Suppose $Y_n = \Gamma_n \backslash \mathbb{H}$ with Γ_n cocompact, torsionfree lattices s.t.

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- 2 Uniform spectral gap for $\Delta \curvearrowright L^2(Y_n)$
- 3 Uniform discreteness

For each Y_n let $\{\psi_j^{(n)}\}$ be ONB of eigenfunctions of $\Delta \curvearrowright L^2(Y_n)$ with associated eigenvalues $\lambda_j^{(n)}$. Let $\mathcal{I} \subset (1/4, \infty)$ be a compact interval. Let $a_n \in L^\infty(Y_n)$ with uniform L^∞ -bound. Then we expect

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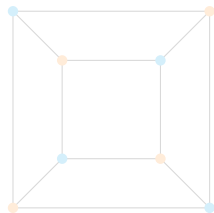
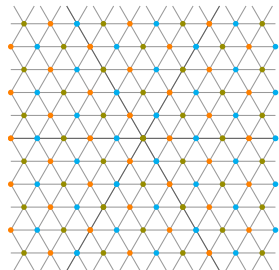
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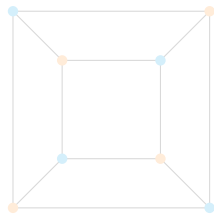
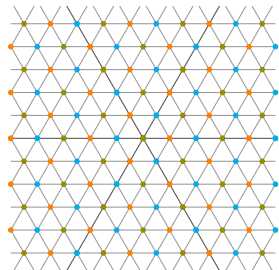
Coloring eigenfunctions

- Sometimes $\Gamma \backslash G/K$ has a non-trivial coloring
- Coloring gives “trivial” coloring eigenfunctions
- Generalization of eigenfunction associated to $-(q+1)$ for regular bipartite graphs



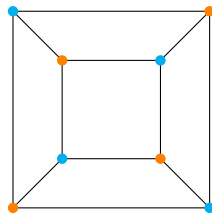
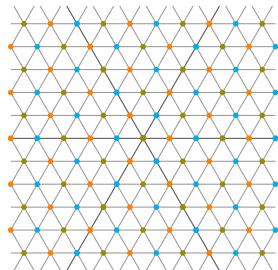
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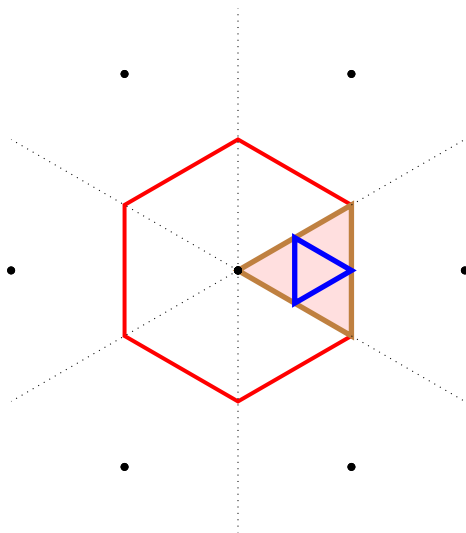


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The tempered spectrum and the exceptional locus



Preceding literature on QE in the BS limit

- Anantharaman-Le Masson '15 - set up the framework for QE in the BS limit and proved it for regular graphs
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Weyl chamber parametrizes relative positions

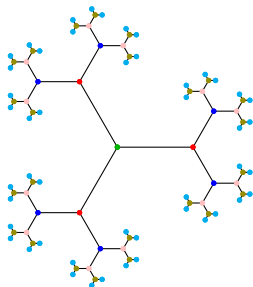


Figure: Half geodesics (Weyl chambers) parametrize relative positions



Metric balls in the tree are polytopal balls

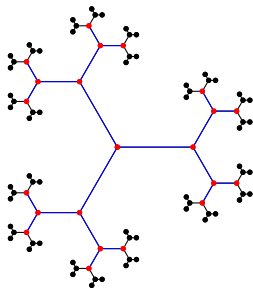
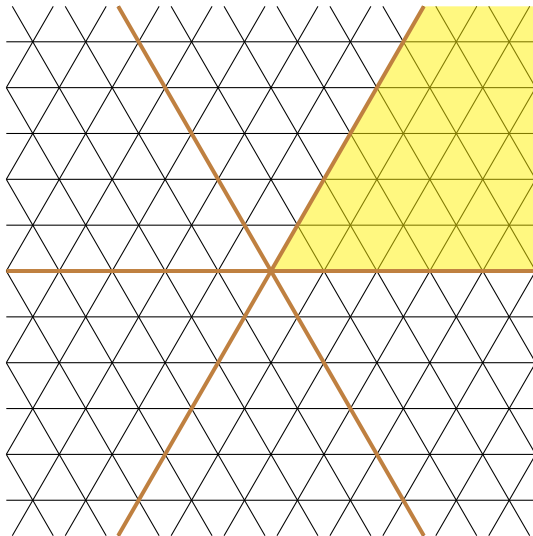


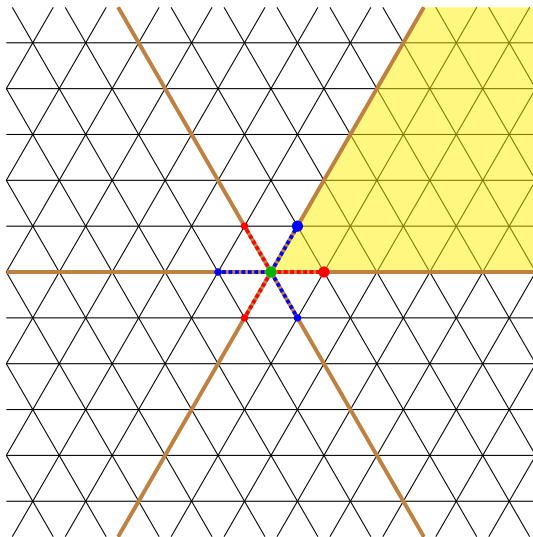
Figure: A polytope (line segment) in the Weyl chamber corresponds to a ball in the tree



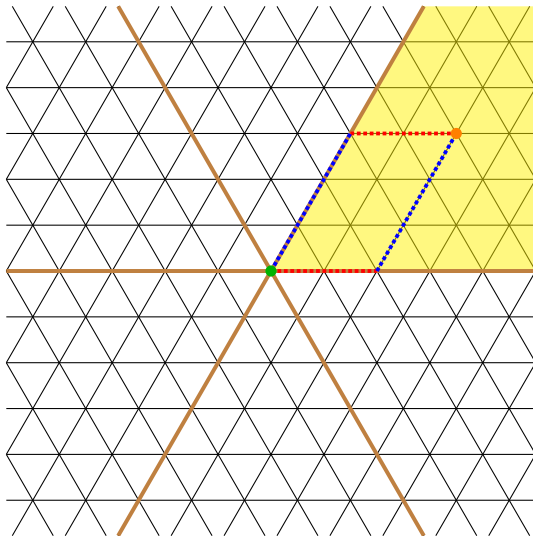
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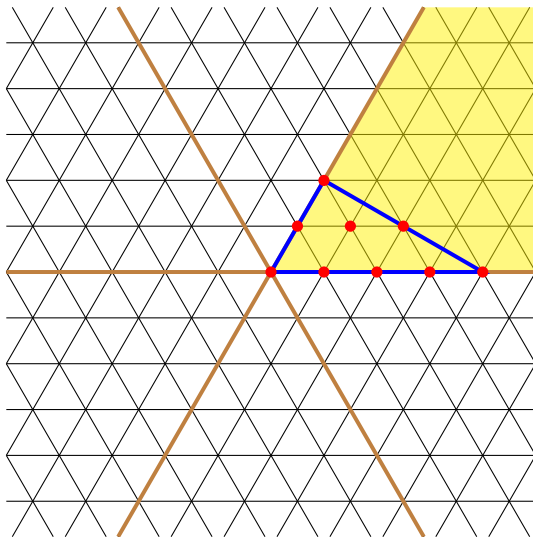
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Polytopes in Weyl chamber define polytopal balls



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Wave propagator to geometric bound

- A_M = wave propagator
 - $U_m \approx$ avg over polytopal ball
 - $P_m(x)$ = polytopal ball centered at x

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$$U_m^* \circ a_n \circ U_m$$

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My method for geometric bound applied to the regular tree

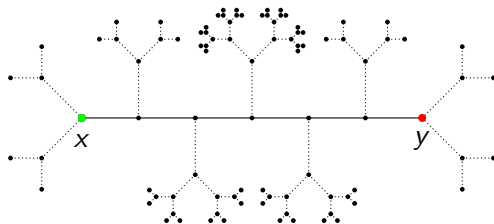


Figure: $P_8(x) \cap P_8(y)$ on 3-regular tree with $d(x, y) = 6$

- Goal: compute volume of intersection (number of vertices)

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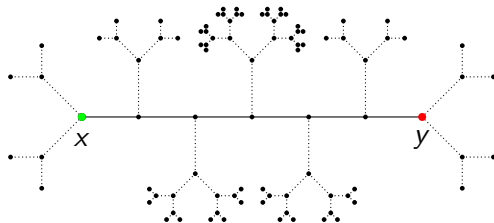
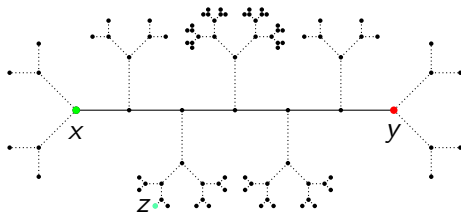


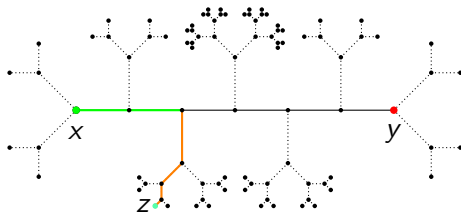
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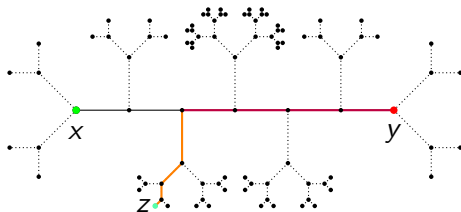
My method (cont.): polytopal parametrization



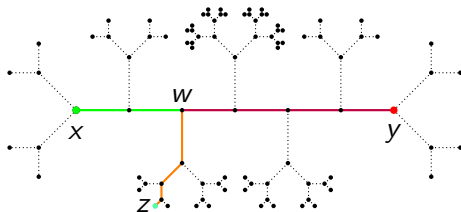
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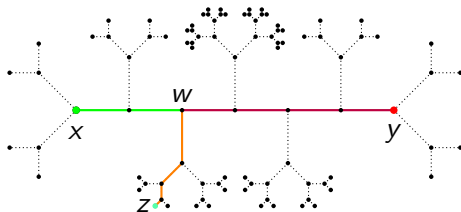
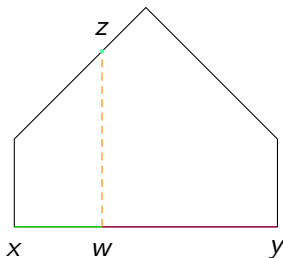
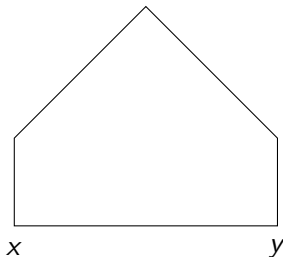
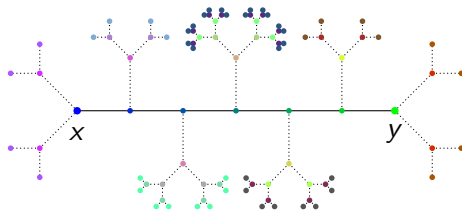


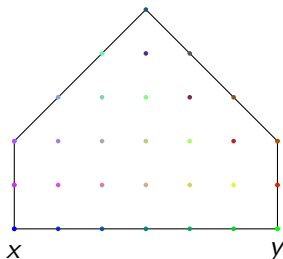
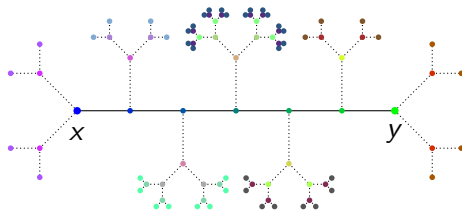
Figure: w is *confluence point*: $\text{geod}(x, w) \cap \text{geod}(y, w) = \{w\}$



My method (cont.): polytopal parametrization



My method (cont.): polytopal parametrization



My method (cont.): Brion's formula

- z gets assigned coordinates (α, β) :

$\alpha = \#$ of steps along geod(x, y) to reach confluence point

$\beta = \#$ of steps from confluence point to z

- $\#\{z \text{ with coordinates } (\alpha, \beta)\} \approx q^\beta = q^{\langle (0,1), (\alpha, \beta) \rangle}$
- Brion's formula for exponential sum over lattice points in polytope:

$$\sum_{\lambda \in Q \cap \mathbb{Z}^n} q^{\langle f, \lambda \rangle} = \sum_{\text{vertices } v \text{ of } Q} C_v q^{\langle f, v \rangle}$$

(in this case $f = (0, 1)$)

- Dominating term comes from v which maximizes $q^{\langle f, v \rangle}$
- C_v are constants depending on f and the cone at vertex v

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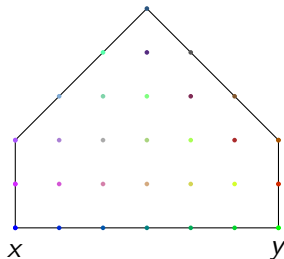
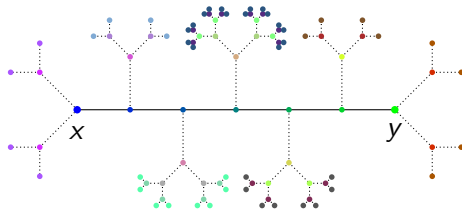
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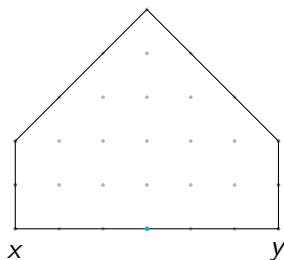
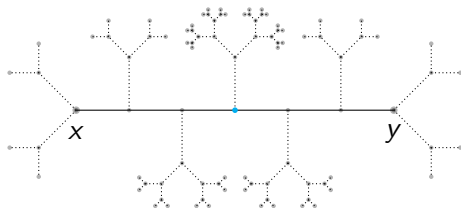
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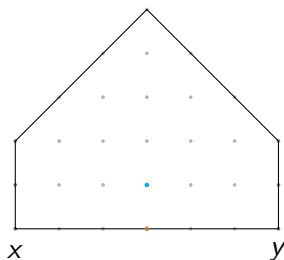
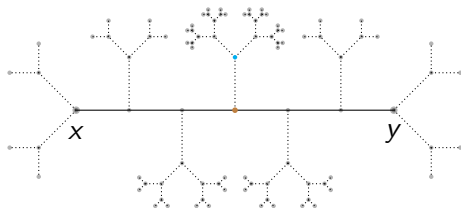
My method (cont.): polytopal parametrization



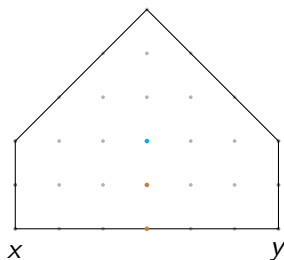
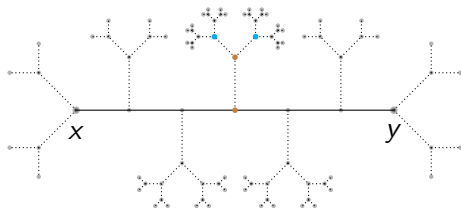
My method (cont.): polytopal parametrization



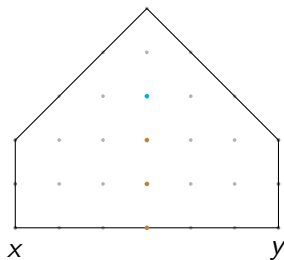
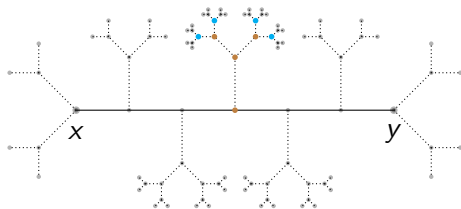
My method (cont.): polytopal parametrization



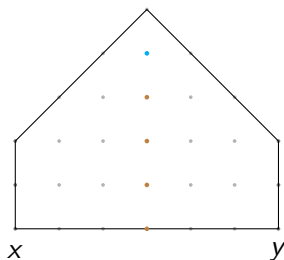
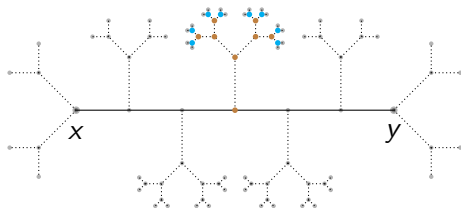
My method (cont.): polytopal parametrization



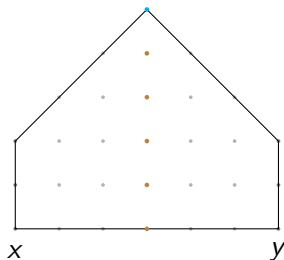
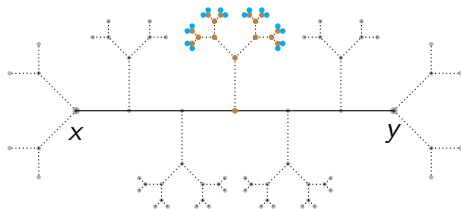
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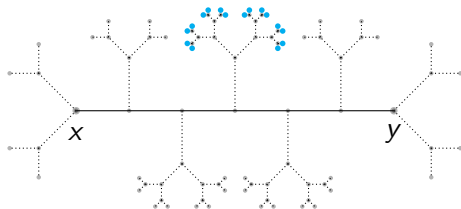
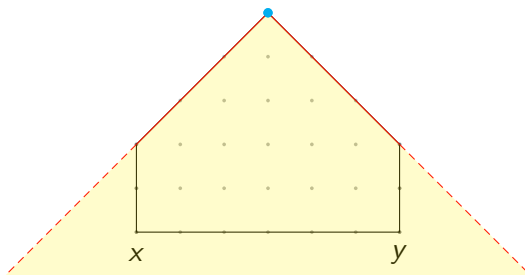


Figure: dominating term corresponds to extremal vertex of polytope



My method (cont.): confluence points

- z gets assigned coordinates (α, β) :

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My method (cont.): confluence points

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(in this case $f = (0, 1)$)

- Dominating term comes from ν which maximizes $q^{\langle f, \nu \rangle}$
- C_ν are constants depending on f and the *cone* at vertex ν

My method (cont.): confluence points for $\text{PGL}(3, F)$

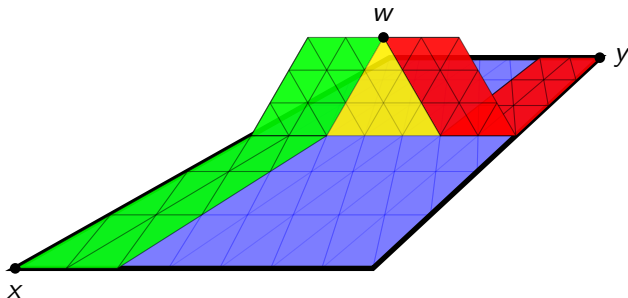


Figure: Confluence points satisfy $\text{para}(x, w) \cap \text{para}(y, w) = \{w\}$

- To set up coordinatization for triples of points in the building, first need to classify confluence points.

My method (cont.): confluence points for $\text{PGL}(3, F)$

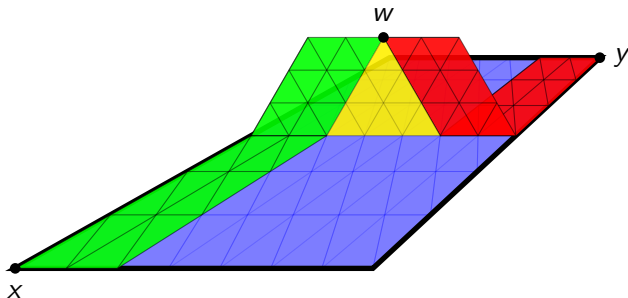


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