QUANTUM ERGODICITY IN THE BENJAMINI-SCHRAMM LIMIT IN HIGHER RANK

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QE in the eigenvalue aspect on hyperbolic surfaces

Theorem (Zelditch '87): Let Y be a compact hyperbolic surface, and let $a \in C^\infty(Y)$. Let $\{\psi_j\}$ be an ONB of Δ on $L^2(Y)$ with eigenvalues $\{\lambda_j\}$. Then

$$\lim_{\lambda\to\infty} \frac{1}{\#\{j : \lambda_j \leq \lambda\}} \sum_{\lambda_j \leq \lambda} \left| \int_Y a \cdot |\psi_j|^2 \, d\mathrm{Vol} - \frac{1}{\mathrm{Vol}(Y)} \int_Y a \, d\mathrm{Vol} \right|^2 = 0.$$

Fix the manifold and vary the spectral window

Interpretation: Generic high energy quantum particles on Y are equidistributed.

Motivation: The geodesic flow on Y is ergodic, so generic classical particles on Y equidistribute.

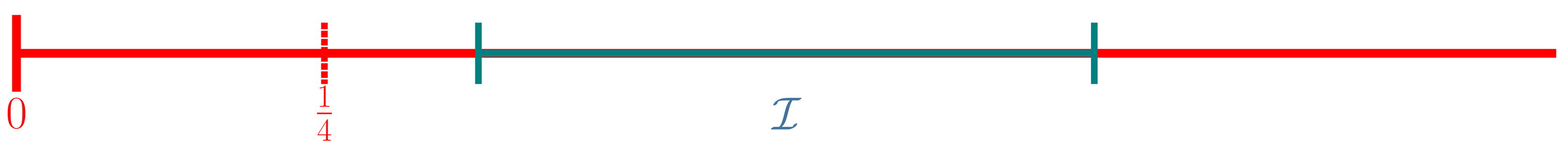
QE in the Benjamini-Schramm limit on hyperbolic surfaces

Definition: A sequence of hyperbolic surfaces (Y_n) **Benjamini-Schramm converges** to $\mathbb{H}$ if asymptotically most points have arbitrarily large injectivity radii, namely that for every $R > 0$:

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(\{y \in Y_n : \mathrm{InjRad}_{Y_n}(y) < R\})}{\mathrm{Vol}(Y_n)} = 0.$$

Theorem (Le Masson-Sahlsten '17): Let (Y_n) be a sequence of compact hyperbolic surfaces. Let $\{\psi_j^{(n)}\}$ be an ONB of eigenfunctions of Δ with eigenvalues $\{\lambda_j^{(n)}\}$. Assume (Y_n) has a uniform spectral gap for Δ, has a universal lower bound on their injectivity radii, and Benjamini-Schramm converges to $\mathbb{H}$. Let $\mathcal{I}$ be a compact subinterval of $(\frac{1}{4}, \infty)$. Let $a_n \in L^\infty(Y_n)$ with a universal L^∞-norm bound. Then

$$\lim_{n\to\infty} \frac{1}{\#\{j : \lambda_j^{(n)} \in \mathcal{I}\}} \sum_{\lambda_j^{(n)} \in \mathcal{I}} \left| \int_{Y_n} a_n \cdot |\psi_j^{(n)}|^2 \, d\mathrm{Vol} - \frac{1}{\mathrm{Vol}(Y_n)} \int_{Y_n} a_n \, d\mathrm{Vol} \right|^2 = 0.$$



Fix the spectral window and vary the manifold

Interpretation: Tempered eigenfunctions on large hyperbolic surfaces are equidistributed on average.

Motivation: The spectrum of Δ on $\mathbb{H}$ is purely absolutely continuous (equal to the interval $[\frac{1}{4}, \infty)$) which is a form of spectral delocalization. If Y_n is "close" to $\mathbb{H}$, its eigenfunctions with eigenvalue in $[\frac{1}{4}, \infty)$ should also exhibit delocalization.

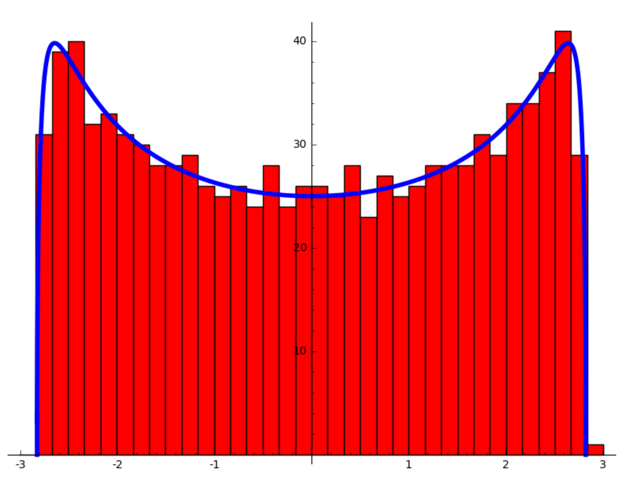
Symmetric spaces

- G = semisimple Lie group without compact factors
- X = Riemannian manifold called **symmetric space**
- K = maximal compact subgroup/stabilizer of pt in X
- $D(G, K)$ = algebra of G-inv. differential ops on X
- Ω = spectrum of $D(G, K)$ acting on $L^2(X)$
- Harish-Chandra isomorphism: $D(G, K)$ is commutative and generated by $\mathrm{rank}(G)$ operators

- $G = \mathrm{SL}(2, \mathbb{R})$
- $X = G/K = \mathbb{H}$
- $K = \mathrm{SO}(2, \mathbb{R})$
- $D(G, K)$ = algebra generated by Δ
- $\Omega = [\frac{1}{4}, \infty)$

	rank one	higher rank
archimedean	hyperbolic surfaces	symmetric spaces
non-archimedean	regular graphs	Bruhat-Tits buildings

Real and p-adic (locally) symmetric spaces



BS convergence implies spectral convergence to Plancherel measure (Weyl law)

Bruhat-Tits buildings

- G = s.s. alg. group over non-archimedean local field F
- $\mathcal{B}$ = simplicial complex called **Bruhat-Tits building**
- $K \approx$ maximal compact subgroup/stabilizer of a vertex in $\mathcal{B}$
- $H(G, K) \approx$ algebra of G-inv. ops on vertices of $\mathcal{B}$ (spherical Hecke algebra)
- Ω = spectrum of $H(G, K)$ acting on $L^2(G/K)$
- Satake isomorphism: $H(G, K)$ is commutative and generated by $\mathrm{rank}(G)$ operators

- $G = \mathrm{PGL}(2, \mathbb{Q}_p)$
- $\mathcal{B}$ = infinite $(p+1)$-regular tree
- $K = \mathrm{PGL}(2, \mathbb{Z}_p)$
- G/K = vertices of the tree
- $H(G, K)$ = algebra generated by adjacency op.
- $\Omega = [-2\sqrt{p}, 2\sqrt{p}]$

Real and p-adic locally symmetric spaces

Suppose $\Gamma < G$ is a cocompact, torsionfree lattice. Then,

$$\Gamma\backslash G/K = \begin{cases} \text{locally symmetric space (e.g. hyperbolic surface)} \\ \text{finite simplicial complex (e.g. finite regular graph),} \end{cases}$$

with universal cover G/K.

Quantum ergodicity in higher rank

Let k be the rank of G. Let A be a maximal split torus in G (e.g. diagonal matrices in $\mathrm{SL}(n)$). Let $M = Z_K(A)$ be the centralizer of A by the maximal compact subgroup K.

Let $\mathcal{C}$ be either $D(G, K)$ or $H(G, K)$. Then $\mathcal{C}$ is generated by k commuting normal operators $C_1, \ldots, C_k$. Thus $L^2(\Gamma\backslash G/K)$ has an ONB of **joint eigenfunctions** $\{\psi_j\}$. By recording the eigenvalue of ψ_j for each C_ℓ as a k-tuple, we obtain the **spectral parameter** ν_j.

	rank one	higher rank
classical	geodesic flow $\mathbb{R} \curvearrowright T^1(\Gamma\backslash\mathbb{H})$	Weyl chamber flow $A \curvearrowright \Gamma\backslash G/M$
quantum	eigenfunctions of Δ on $L^2(\Gamma\backslash\mathbb{H})$	joint eigenfunctions of $\mathcal{C} \curvearrowright L^2(\Gamma\backslash G/K)$

Quantization of higher rank ergodic flow ($\mathbb{R}^k$ or $\mathbb{Z}^k$ action)

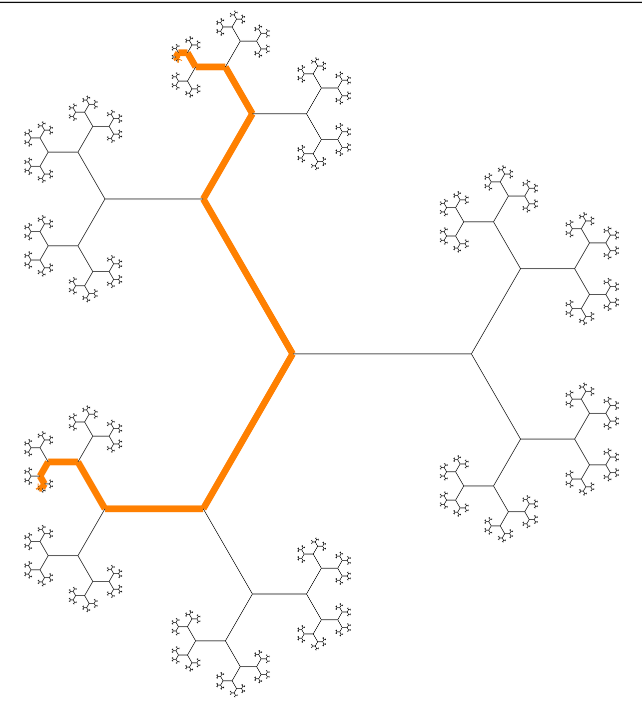
In the non-archimedean case, Γ may preserve a an r-coloring, whence we obtain r **coloring eigenfunctions**. This is a generalization of $-(q+1)$ as an eigenvalue for bipartite graphs.

Main result

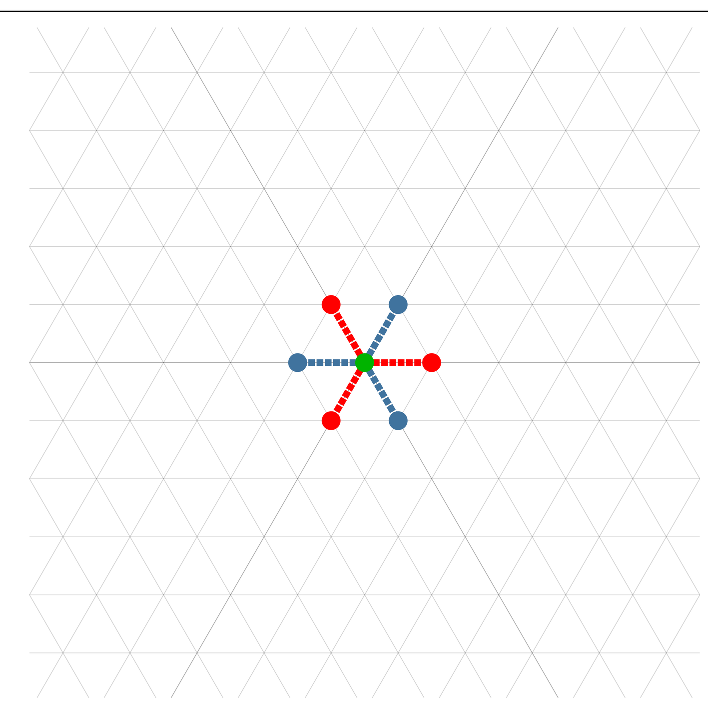
Theorem (P. '23) Let $G = \mathrm{PGL}(3, F)$ and $K = \mathrm{PGL}(3, \mathcal{O})$, where F is a non-archimedean local field of arbitrary characteristic, and $\mathcal{O}$ is its ring of integers. Let $\Gamma_n < G$ be a sequence of torsionfree lattices, and let $Y_n = \Gamma_n\backslash G/K$. Suppose $\mathrm{card}(Y_n) \to \infty$. Let $\Theta \subset \Omega$ be a compact subset with positive Plancherel measure and not meeting a certain codimension one sublocus Ξ. Let $\{\psi_j^{(n)}\}$ be an ONB of eigenfunctions of $H(G, K)$ acting on $L^2(Y_n)$ with spectral parameters $\{\nu_j^{(n)}\}$. Let $a_n \in L^\infty(Y_n)$ with universal L^∞-norm bound and orthogonal to all non-trivial coloring eigenfunctions. Then

$$\lim_{n\to\infty} \frac{1}{\#\{j : \nu_j^{(n)} \in \Theta\}} \sum_{\psi_j^{(n)} : \nu_j^{(n)} \in \Theta} \left| \sum_{v \in Y_n} a_n(v) \cdot |\psi_j^{(n)}(v)|^2 - \frac{1}{\mathrm{card}(Y_n)} \sum_{v \in Y_n} a_n(v) \right|^2 = 0.$$

Geometry of Bruhat-Tits buildings



Apartments in the tree are bi-infinite geodesics



An apartment in the building for $\mathrm{PGL}(3)$

Some remarks about the proof

- Anantharaman-Le Masson '15 introduced QE in the BS limit (for regular graphs). Their proof involved "microlocal analysis on regular trees". Brooks-Le Masson-Lindenstrauss '16 found a new proof using "**wave propagation**" on regular graphs. The case of arbitrary graphs was treated by Anantharaman-Sabri '19.
- The wave propagator roughly corresponds to averaging over "balls" of different radii. It has been adapted for hyperbolic surfaces by Le Masson-Sahlsten '17, rank one locally symmetric spaces by Abert-Bergeron-Le Masson '18, and locally symmetric spaces associated to $\mathrm{SL}(d, \mathbb{R})$ by Brumley-Matz '21. Brumley-Matz '21 has a mistake in the "**geometric bound**".
- In rank one metric balls suffice; in higher rank one must use "**polytopal balls**".
- The wave propagator has desirable spectral properties, ultimately allowing one to reduce to bounding the norm of the kernel function of the wave propagator.
- BS convergence allows one to lift to analyzing the kernel function on G/K.
- After changing variables, one may bound the norm of the kernel function using an **ergodic theorem** from Nevo '98, which bounds the op. norm of convolution ops associated to ergodic actions of semisimple algebraic groups over local fields.
- An input to the Nevo ergodic theorem is the volume of the set defining the convolution. One is thus led to bounding the volume of the intersection of "polytopal balls" in G/K (the "geometric bound"). **This is the hardest step**.

Work in progress

Theorem* (Brumley-Marshall-Matz-P. '24+) Let Y_n be a sequence of compact quotients of $\mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$ with $\mathrm{Vol}(Y_n) \to \infty$. Let $\Theta \subset \Omega$ be a compact subset with positive Plancherel measure and not meeting a certain codimension one sublocus Ξ. Let $\{\psi_j^{(n)}\}$ be an ONB of eigenfunctions of $D(G, K)$ acting on $L^2(Y_n)$ with spectral parameters $\{\nu_j^{(n)}\}$. Let $a_n \in L^\infty(Y_n)$ with universal L^∞-norm bound. Then

$$\lim_{n\to\infty} \frac{1}{\#\{j : \nu_j^{(n)} \in \Theta\}} \sum_{\psi_j^{(n)} : \nu_j^{(n)} \in \Theta} \left| \int_{Y_n} a_n \cdot |\psi_j^{(n)}|^2 d\mathrm{Vol} - \frac{1}{\mathrm{Vol}(Y_n)} \int_{Y_n} a_n d\mathrm{Vol} \right|^2 = 0.$$

The technique seems to also work for all symmetric spaces except those of type F_4 and G_2.