Quantum ergodicity on the Bruhat-Tits building for PGL(3, F) in the Benjamini-Schramm limit

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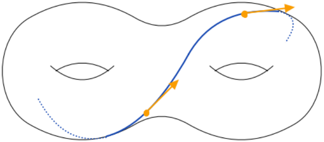
Geodesic flow on hyperbolic surface

- Y compact hyperbolic surface
- $\Phi_t \curvearrowright T^1 Y$ geodesic flow



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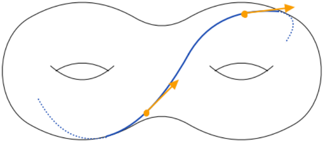

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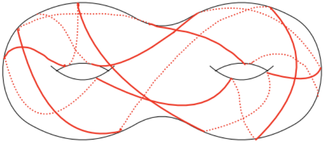

Classical and quantum mechanics on Y

$$\begin{tikzcd}
\text{classical mechanics} \approx \underset{\text{geodesic flow}}{\Phi_t \curvearrowright T^1 Y} \arrow[d, bend right, "\text{quantization}"'] \\
\text{quantum mechanics} \approx \underset{\text{Schrödinger flow}}{e^{ith\Delta} \curvearrowright L^2(Y)} \arrow[u, bend right, dashed, "h \to 0"']
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Quantum particles

- Renormalize volume measure: $\text{đVol} = \frac{d\text{Vol}}{\text{Vol}(Y)}$
- Quantum particle $\rightsquigarrow \psi \in L^2(Y, \text{đVol})$ with $||\psi||_2 = 1$

$$\begin{aligned} \mathbb{P}(\text{observing } \psi \text{ in } E \subset Y) &= \int_E |\psi|^2 \, \text{đVol} \\ &= \int_Y \mathbb{1}_E \cdot |\psi|^2 \, \text{đVol} \end{aligned}$$

- If ψ were *equidistributed*:

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The Laplacian

- Eigendata of Δ:

$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ eigenvalues of Δ

$\{\psi_j\}$ ONB of eigenfunctions of Δ

- In QM, ψ_j has energy $h^2\lambda_j$. Let $h_j = \frac{1}{\sqrt{\lambda_j}}$.

fix h and let $\lambda_j \to \infty$ $\approx$ fix energy and let $h_j \to 0$

- As $\lambda_j \to \infty$, should "recover" ergodicity $\rightsquigarrow$ ψ_j *equidistributes*

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Theorem (QE in the large eigenvalue limit; Snirelman, Zelditch, Colin de Verdiere)

Let $a \in C^\infty(Y)$. *Then*

$$\lim_{\lambda \to \infty} \frac{1}{\#\{j : \lambda_j \leq \lambda\}} \sum_{j : \lambda_j \leq \lambda} \left| \int_Y a \cdot |\psi_j|^2 \, \text{đVol} - \int_Y a \, \text{đVol} \right|^2 = 0.$$

- Average over eigenfunctions with eigenvalue less than λ
- Compare the measures $|\psi_j|^2\text{đVol}$ and đVol weakly (integrate against test function)
- Interpretations:
    1. Generic high energy quantum particles equidistribute.
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Visualization of quantum ergodicity

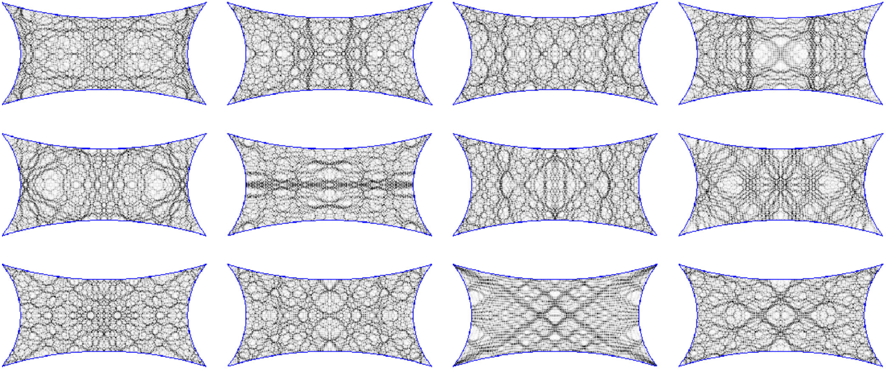

Figure: Image made by Alex Barnett

QE in large eigenvalue limit vs. QE in the BS limit

- Eigenvalues of Δ lie in $[0, \infty)$
- QE in the large eigenvalue limit:

fix the manifold & vary the spectral window

- QE in the Benjamini-Schramm limit:

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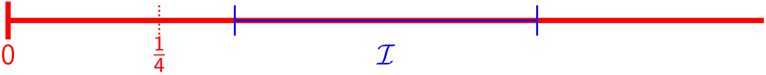

Benjamini-Schramm convergence

(Y_n) *Benjamini-Schramm converges* to $\mathbb{H}$ if, for every $R > 0$,

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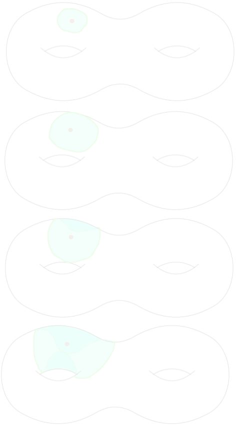

Interpretation: most points have arbitrarily large injectivity radius

Spectrum of Δ on $\mathbb{H}$ is $[\frac{1}{4}, \infty)$.

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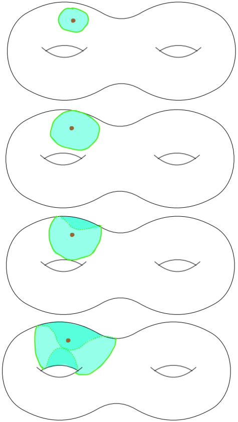

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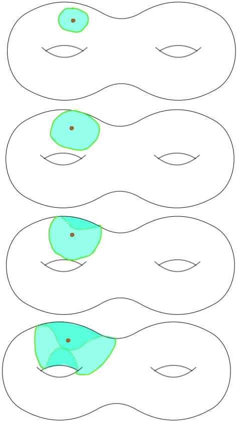

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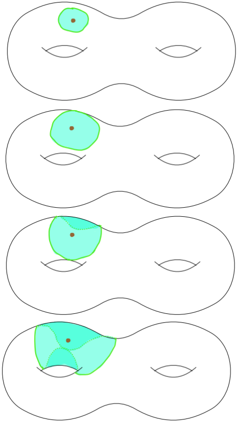

QE in the BS limit for hyperbolic surfaces

Theorem (Le Masson-Sahlsten '17)

Suppose (Y_n) *is a sequence of compact hyperbolic surfaces s.t.*

1. *Benjamini-Schramm convergence:* $Y_n \xrightarrow{BS} \mathbb{H}$.
2. *Uniform spectral gap:* $\lambda_1^{(n)}$ *bounded away from 0 for all n.*
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Let $\{\psi_j^{(n)}\}$ *be ONB of eigenfunctions for* Δ *acting on* $L^2(Y_n)$ *with eigenvalues* $0 = \lambda_0^{(n)} \leq \lambda_1^{(n)} \leq \dots$. *Let* $\mathcal{I} \subset (\frac{1}{4}, \infty)$ *be a compact subinterval. Let* $a_n \in L^\infty(Y_n)$ *with uniformly bounded* L^∞*-norm. Then*

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	rank one	higher rank
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Quantization in higher rank

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higher rank → geod. flow NOT ergodic (BUT ergodic $\mathbb{R}^k$-action) → QE involves k operators

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- G = semisimple Lie group over $\mathbb{R}$ (w/o compact factors)
- X is associated Riemannian manifold called *symmetric space*
- $X = G/K$ with K a maximal compact subgroup
- $D(G, K)$ = G-invariant differential operators on X
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$D(G, K)$ = algebra generated by Δ

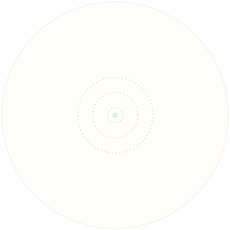

Figure: Δ closely related to averaging over spheres in $\mathbb{H}$

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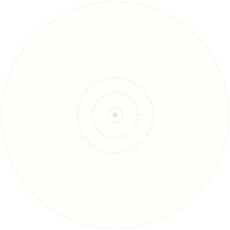

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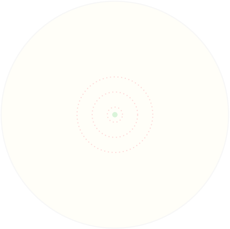

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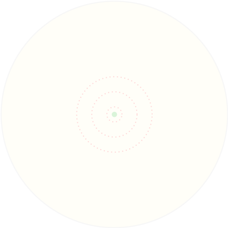

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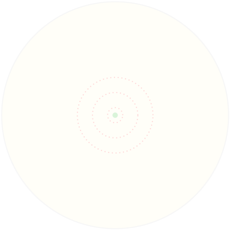

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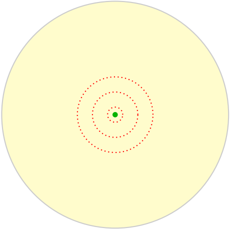

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- G = semisimple algebraic group over F (non-archimedean local field)
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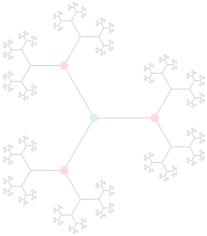

Figure: Adjacency operator $\mathcal{A}$ on tree involves summing over sphere of radius 1

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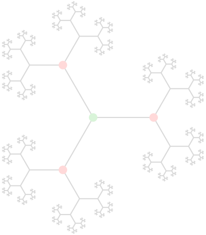

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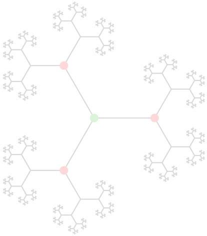

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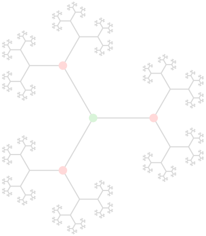

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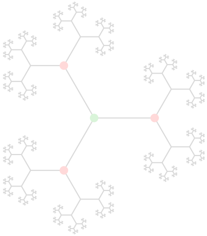

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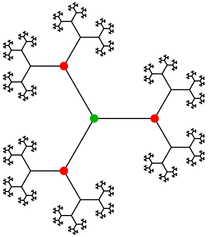

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Buildings are composed of branching apartments

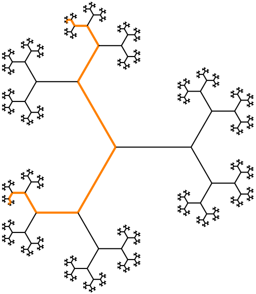
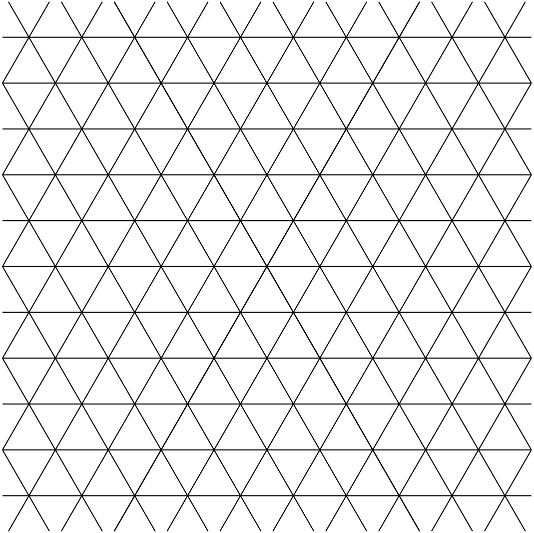


Figure: An apartment in the tree is a bi-infinite geodesic.

An apartment in the Bruhat-Tits building of PGL$(3, F)$



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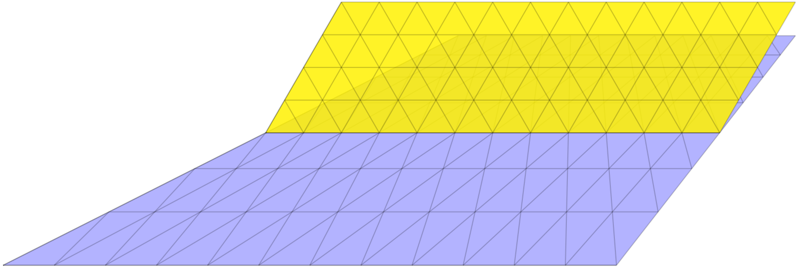
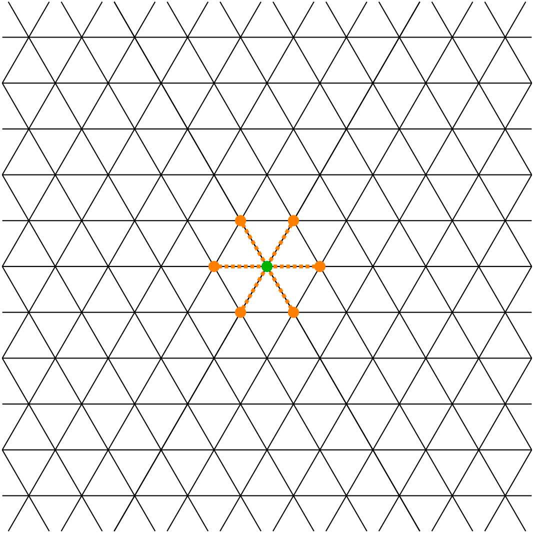


$H(G, K)$ generated by refinements of adjacency operator



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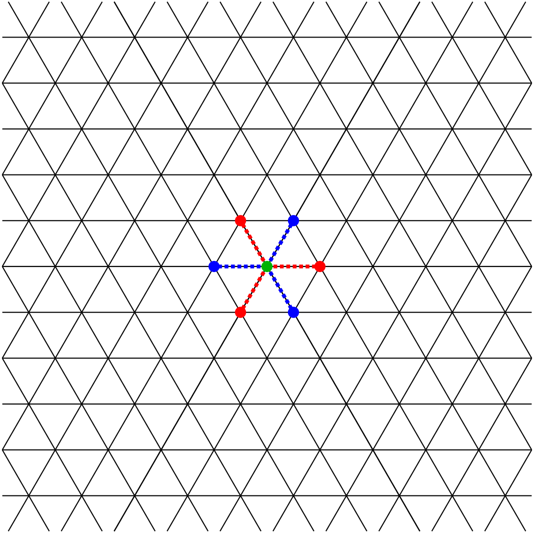

Quotients of X and $\mathcal{B}$

- $\Gamma < G$ cocompact, torsionfree lattice

$$\Gamma\backslash G/K \text{ is } \begin{cases} \text{locally symmetric space (e.g. hyperbolic surface)} \\ \text{finite simplicial complex (e.g. finite regular graph)} \end{cases}$$

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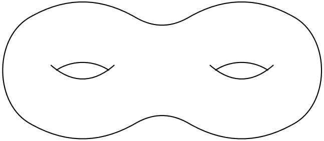

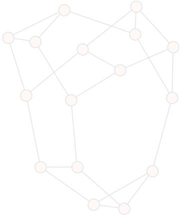

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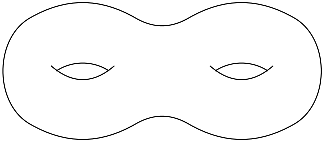

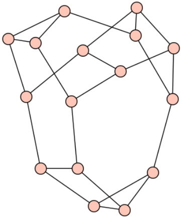

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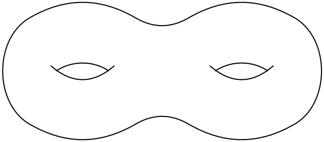

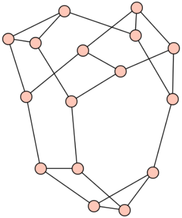

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Joint eigenfunctions and spectral parameters

- Let $\mathcal{C} =$ either $D(G, K)$ or $H(G, K)$
- $\mathcal{C}$ generated by k operators $A_1, \ldots, A_k$

$$\mathcal{C} \curvearrowright L^2(\Gamma \backslash G / K) = \bigoplus_j \mathbb{C}\psi_j \quad \text{(joint eigenfunctions)}$$

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Figure: Ω^{+}_{temp} for Δ on $\mathbb{H}$ is $[1/4, \infty)$

Figure: Ω^{+}_{temp} for $\mathcal{A}$ on $(q+1)$-regular tree is $[-2\sqrt{q}, 2\sqrt{q}]$

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Figure: Ω^+_{temp} for Δ on $\mathbb{H}$ is $[1/4, \infty)$

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BS convergence implies Plancherel convergence

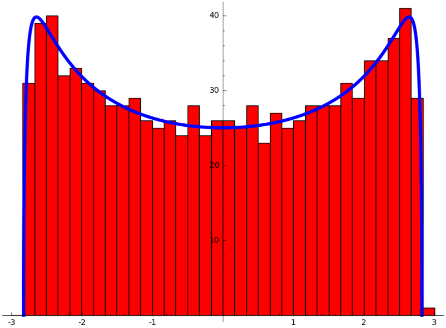

Figure: Distribution of eigenvalues for large random 3-regular graph

$$\frac{\#\{j : \lambda_j^{(n)} \in \mathcal{I}\}}{\mathrm{Vol}(Y_n)} \to \mu(\mathcal{I})$$

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3. ~~Uniform discreteness~~ follows from discreteness of building

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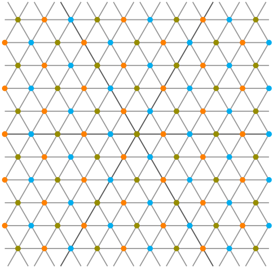
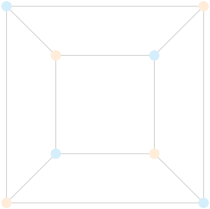
Coloring eigenfunctions

- Sometimes $\Gamma\backslash G/K$ has a non-trivial coloring

- Coloring gives "trivial" coloring eigenfunctions

- Generalization of eigenfunction associated to $-(q+1)$ for regular bipartite graphs





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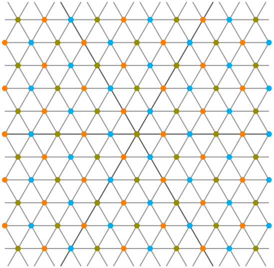

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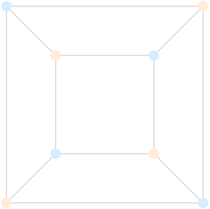

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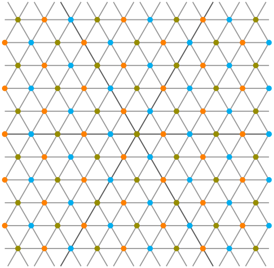

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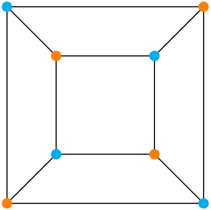
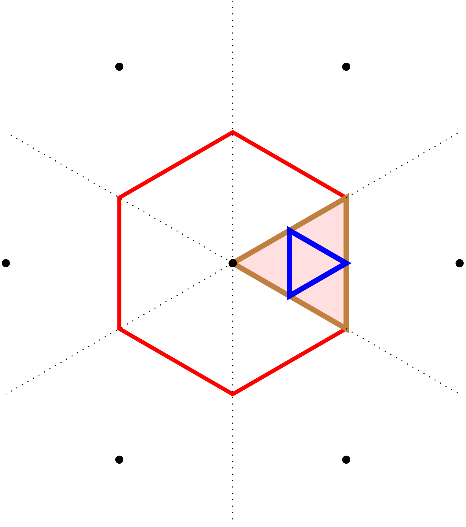


The tempered spectrum and the exceptional locus



Preceding literature on QE in the BS limit

- Anantharaman-Le Masson '15 - set up the framework for QE in the BS limit and proved it for regular graphs
- Brooks-Le Masson-Lindenstrauss '16 - reproved QE in the BS limit for regular graphs using wave propagator method
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wave propagator

(polytopal ball averaging operators)

↓

geometric bound

(intersections of polytopal balls)

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↓

geometric bound

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Wave propagator to geometric bound

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$\downarrow$

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wave propagator

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↓

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Weyl chamber parametrizes relative positions

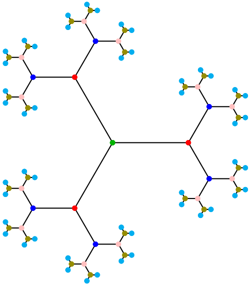


Figure: Half geodesics (Weyl chambers) parametrize relative positions

Metric balls in the tree are polytopal balls

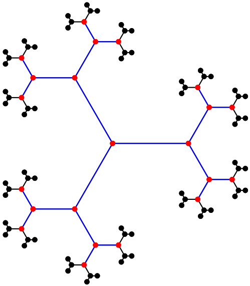
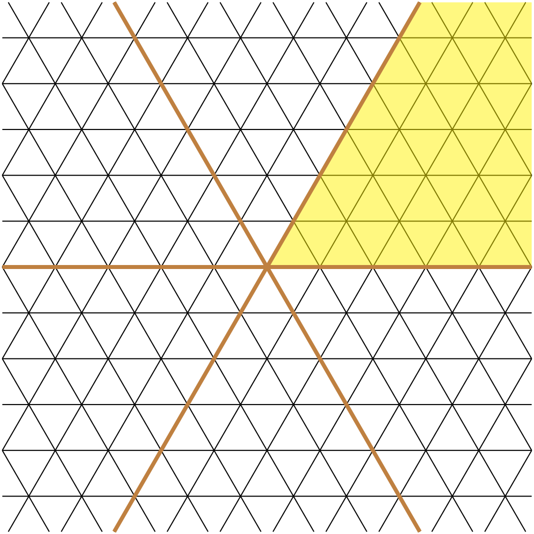


Figure: A polytope (line segment) in the Weyl chamber corresponds to a ball in the tree

Weyl chamber parametrizes relative positions



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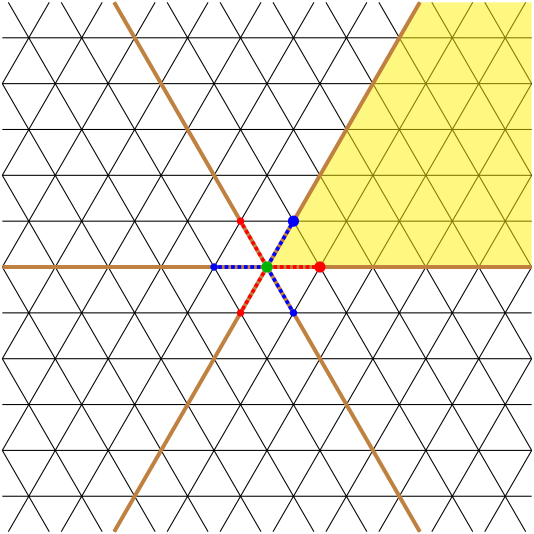

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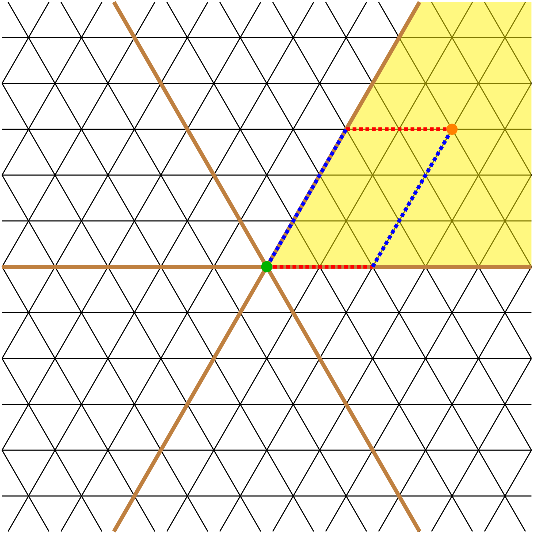
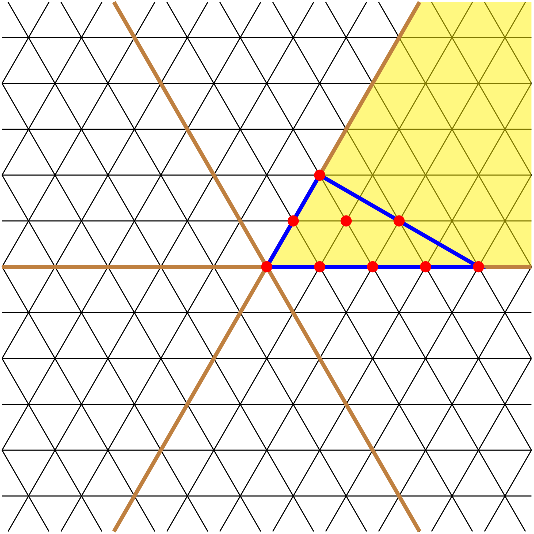


Polytopes in Weyl chamber define polytopal balls



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Wave propagator to geometric bound

- A_M = wave propagator
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  - $P_m(x)$ = polytopal ball centered at x

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Analyze kernel function

$$U_m^* \circ a_n \circ U_m$$

$$\downarrow$$

integrate a_n over $P_m(x) \cap P_m(y)$

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convolution op. assoc.

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Ergodic theorem of Nevo

$$\begin{array}{c}\text{norm of} \\ \text{conv. op.}\end{array} \lesssim \frac{1}{\mathrm{Vol}\left(P_m(x) \cap P_m(y)\right)^{\delta}}$$

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Ergodic theorem of Nevo

$$\text{norm of conv. op.} \lesssim \frac{1}{\mathrm{Vol}\big(P_m(x) \cap P_m(y)\big)^\delta}$$

Wave propagator to geometric bound

- A_M = wave propagator
  - $U_m \approx$ avg over polytopal ball
  - $P_m(x)$ = polytopal ball centered at x

$$\sum \left| \int_{Y_n} a_n \cdot |\psi_j^{(n)}|^2 \, đ\mathrm{Vol} \right|^2 = \sum \left| \langle \psi_j^{(n)}, a_n \cdot \psi_j^{(n)} \rangle \right|^2 \lesssim \sum_{\text{all } \psi_j^{(n)}} ||A_M \psi_j^{(n)}||^2$$

$$\boxed{\text{Analyze kernel function}}$$

$$U_m^* \circ a_n \circ U_m$$

$$\downarrow$$

integrate a_n over $P_m(x) \cap P_m(y)$

$$\downarrow$$

convolution op. assoc.

to $P_m(x) \cap P_m(y)$

$$\boxed{\text{Ergodic theorem of Nevo}}$$

$$\begin{matrix}\text{norm of} \\ \text{conv. op.}\end{matrix} \lesssim \frac{1}{\mathrm{Vol}\big(P_m(x) \cap P_m(y)\big)^{\delta}}$$

My method for geometric bound applied to the regular tree

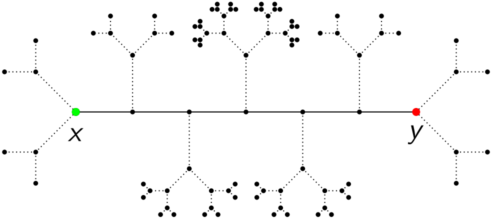

Figure: $P_8(x) \cap P_8(y)$ on 3-regular tree with $d(x, y) = 6$

- Goal: compute volume of intersection (number of vertices)

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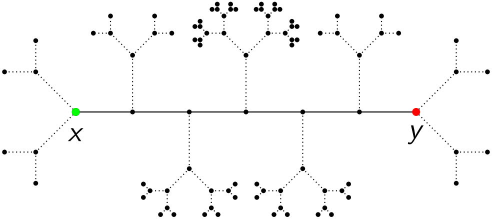

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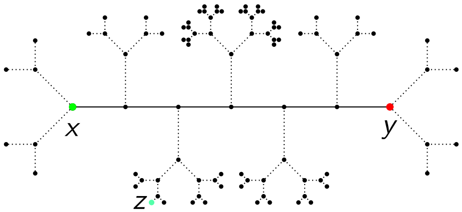
My method (cont.): polytopal parametrization



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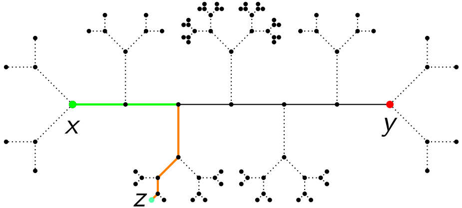

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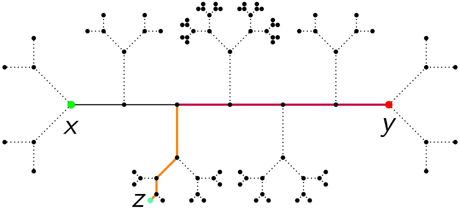

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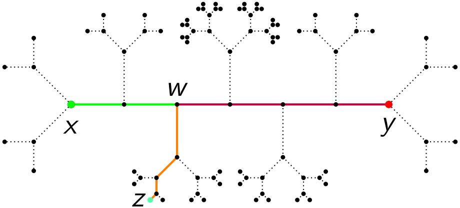

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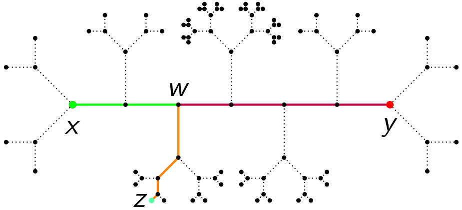
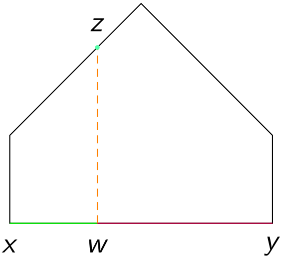


Figure: w is *confluence point*: $\mathrm{geod}(x, w) \cap \mathrm{geod}(y, w) = \{w\}$



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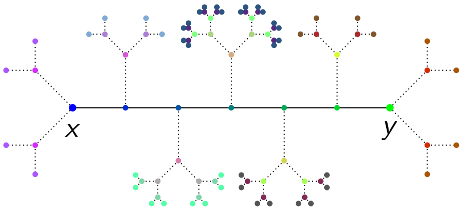

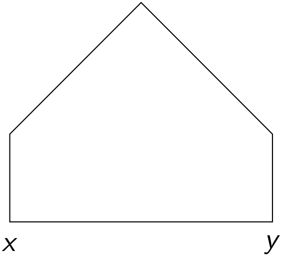

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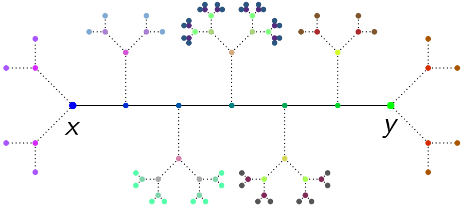

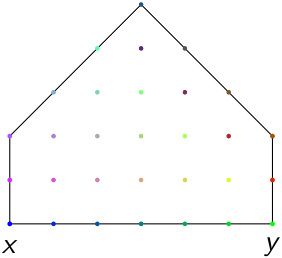

My method (cont.): Brion's formula

- *z* gets assigned coordinates (α, β):

$$\alpha = \# \text{ of steps along geod}(x, y) \text{ to reach confluence point}$$
$$\beta = \# \text{ of steps from confluence point to } z$$

- $\#\{z \text{ with coordinates } (\alpha, \beta)\} \approx q^{\beta} = q^{\left\langle (0,1), (\alpha, \beta) \right\rangle}$
- Brion's formula for exponential sum over lattice points in polytope:

$$\sum_{\lambda \in Q \cap \mathbb{Z}^n} q^{\langle f, \lambda \rangle} = \sum_{\text{vertices } v \text{ of } Q} C_v q^{\langle f, v \rangle}$$

$$\Big(\text{in this case } f = (0, 1)\Big)$$

  - Dominating term comes from v which maximizes $q^{\langle f, v \rangle}$
  - C_v are constants depending on f and the *cone* at vertex v

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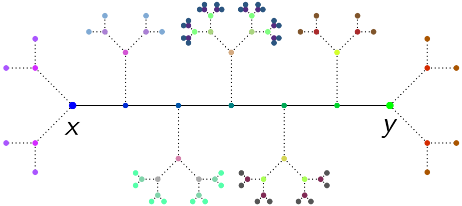
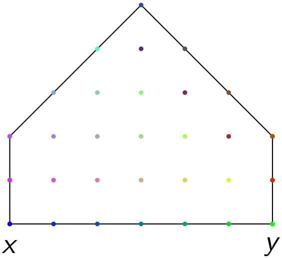
My method (cont.): polytopal parametrization





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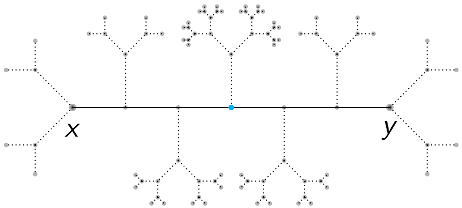

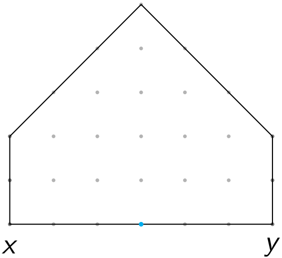

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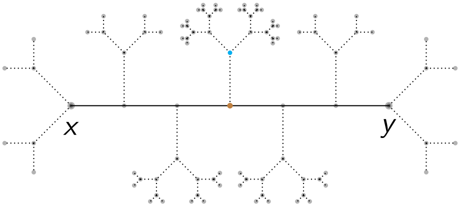

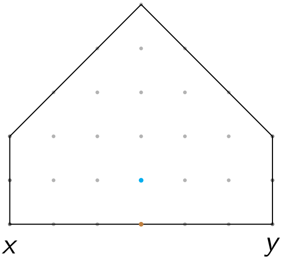

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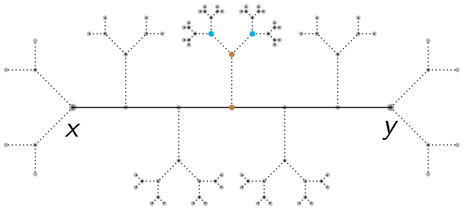

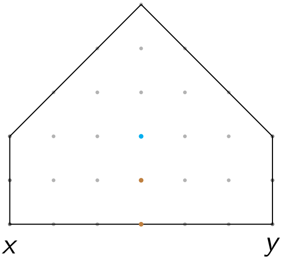

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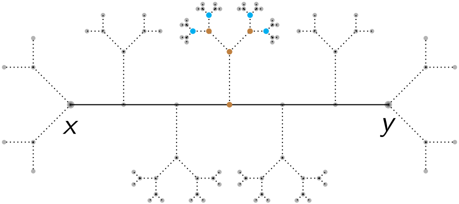

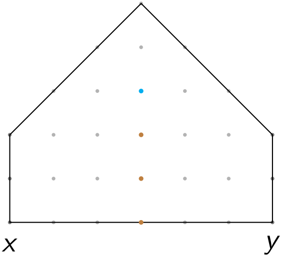

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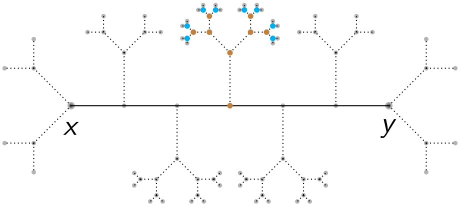

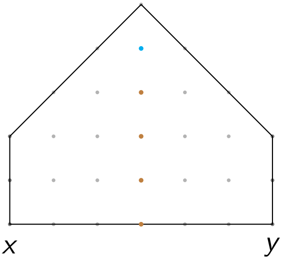

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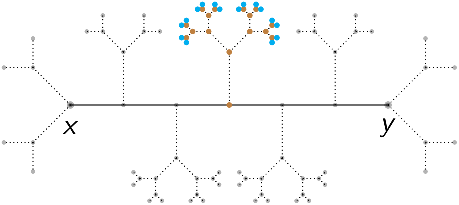

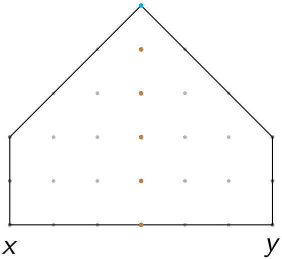

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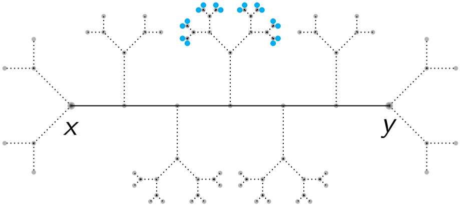
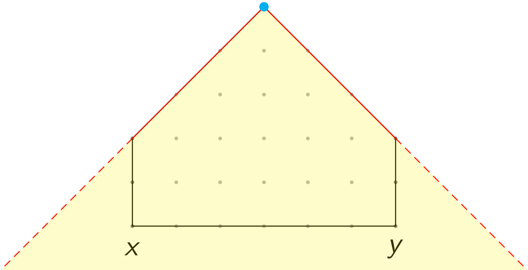


Figure: dominating term corresponds to extremal vertex of polytope



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My method (cont.): confluence points for $\mathrm{PGL}(3, F)$

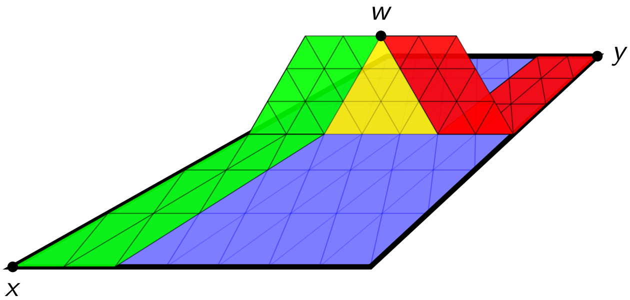
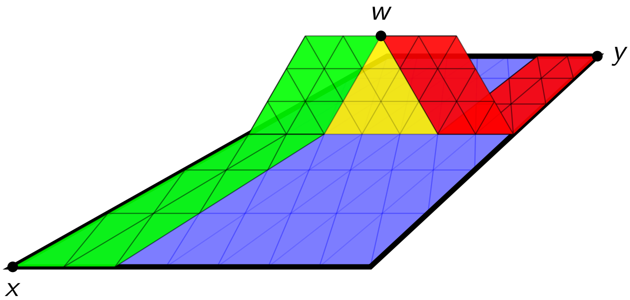


Figure: Confluence points satisfy $\mathrm{para}(x, w) \cap \mathrm{para}(y, w) = \{w\}$

- To set up coordinatization for triples of points in the building, first need to classify confluence points.

My method (cont.): confluence points for $\mathrm{PGL}(3, F)$



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