QUANTUM ERGODICITY ON THE BRUHAT-TITS BUILDING FOR PGL$(3, F)$ IN THE BENJAMINI-SCHRAMM LIMIT

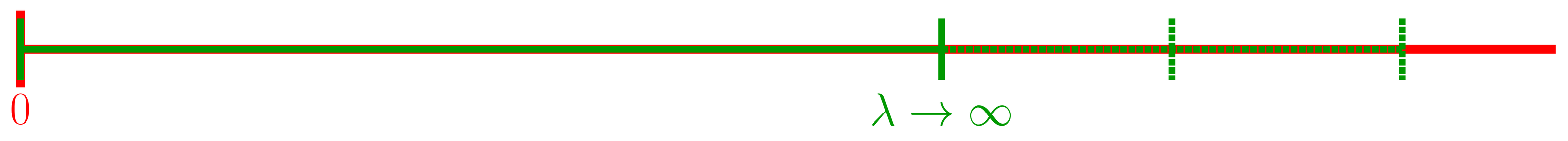
Carsten Peterson (Aalto University and Paderborn University)

Preprint at arXiv:2304.08641

QE in the large eigenvalue limit on hyperbolic surfaces

Theorem (Zelditch '87): Let Y be a compact hyperbolic surface, and let $a \in C^\infty(Y)$. Let $\{\psi_j\}$ be an ONB of Δ on $L^2(Y)$ with eigenvalues $\{\lambda_j\}$. Then

$$\lim_{\lambda\to\infty} \frac{1}{\#\{j : \lambda_j \le \lambda\}} \sum_{\lambda_j \le \lambda} \left| \int_Y a \cdot |\psi_j|^2 \, d\mathrm{Vol} - \frac{1}{\mathrm{Vol}(Y)} \int_Y a \, d\mathrm{Vol} \right|^2 = 0.$$



Fix the manifold and vary the spectral window

Interpretation: Generic high energy quantum particles on Y are equidistributed.

Motivation: The geodesic flow on Y is ergodic, so generic classical particles on Y equidistribute.

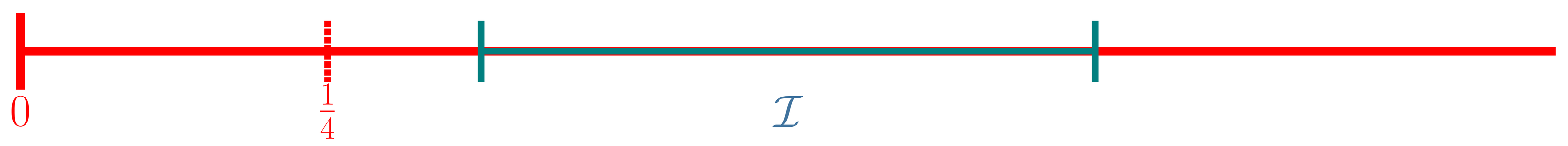
QE in the Benjamini-Schramm limit on hyperbolic surfaces

Definition: A sequence of hyperbolic surfaces (Y_n) **Benjamini-Schramm converges** to $\mathbb{H}$ if asymptotically most points have arbitrarily large injectivity radii, namely that for every $R > 0$:

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(\{y \in Y_n : \mathrm{InjRad}_{Y_n}(y) < R\})}{\mathrm{Vol}(Y_n)} = 0.$$

Theorem (Le Masson-Sahlsten '17): Let (Y_n) be a sequence of compact hyperbolic surfaces. Let $\{\psi_j^{(n)}\}$ be an ONB of eigenfunctions of Δ with eigenvalues $\{\lambda_j^{(n)}\}$. Assume (Y_n) has a uniform spectral gap for Δ, has a universal lower bound on their injectivity radii, and Benjamini-Schramm converges to $\mathbb{H}$. Let $\mathcal{I}$ be a compact subinterval of $(\frac{1}{4}, \infty)$. Let $a_n \in L^\infty(Y_n)$ with a universal L^∞-norm bound. Then

$$\lim_{n\to\infty} \frac{1}{\#\{j : \lambda_j^{(n)} \in \mathcal{I}\}} \sum_{\lambda_j^{(n)} \in \mathcal{I}} \left| \int_Y a_n \cdot |\psi_j^{(n)}|^2 \, d\mathrm{Vol} - \frac{1}{\mathrm{Vol}(Y_n)} \int_{Y_n} a_n \, d\mathrm{Vol} \right|^2 = 0.$$



Fix the spectral window and vary the manifold

Interpretation: Tempered eigenfunctions on large hyperbolic surfaces are equidistributed on average.

Motivation: The spectrum of Δ on $\mathbb{H}$ is purely absolutely continuous (equal to the interval $[\frac{1}{4}, \infty)$) which is a form of spectral delocalization. If Y_n is "close" to $\mathbb{H}$, its eigenfunctions with eigenvalue in $[\frac{1}{4}, \infty)$ should also exhibit delocalization.

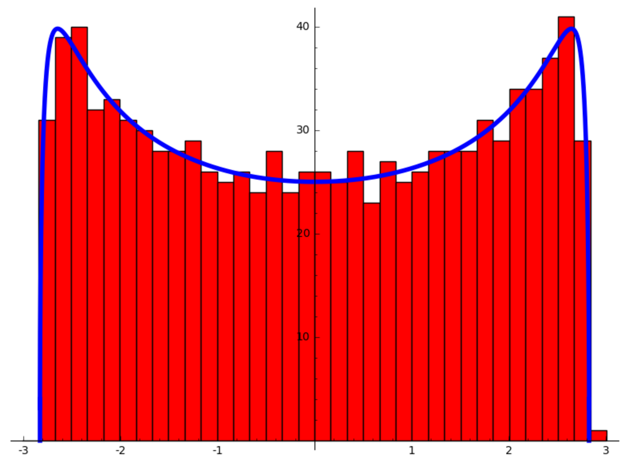
Symmetric spaces

- G = semisimple Lie group without compact factors
- X = Riemannian manifold called **symmetric space**
- K = maximal compact subgroup/stabilizer of pt in X
- $D(G, K)$ = algebra of G-inv. differential ops on X
- Ω = spectrum of $D(G, K)$ acting on $L^2(X)$
- $D(G, K)$ is commutative and freely generated by $\mathrm{rank}(G)$ operators

- $G = \mathrm{SL}(2, \mathbb{R})$
- $X = \mathbb{H}$
- $K = \mathrm{SO}(2, \mathbb{R})$
- $D(G, K)$ = algebra generated by Δ
- $\Omega = [\frac{1}{4}, \infty)$

	rank one	higher rank
archimedean	hyperbolic surfaces	symmetric spaces
non-archimedean	regular graphs	Bruhat-Tits buildings

Real and p-adic (locally) symmetric spaces



BS convergence implies spectral convergence to Plancherel measure

Bruhat-Tits buildings

- G = s.s. alg. group over non-archimedean local field F
- $\mathcal{B}$ = simplicial complex called **Bruhat-Tits building**
- $K \approx$ maximal compact subgroup/stabilizer of a vertex in $\mathcal{B}$
- $H(G, K) \approx$ algebra of G-inv. ops on vertices of $\mathcal{B}$
- Ω = spectrum of $H(G, K)$ acting on $L^2(G/K)$
- $H(G, K)$ is commutative and freely generated by $\mathrm{rank}(G)$ operators

- $G = \mathrm{PGL}(2, \mathbb{Q}_p)$
- $\mathcal{B}$ = infinite $(p+1)$-regular tree
- $K = \mathrm{PGL}(2, \mathbb{Z}_p)$
- G/K = vertices of the tree
- $H(G, K)$ = algebra generated by adjacency op.
- $\Omega = [-2\sqrt{p}, 2\sqrt{p}]$

Joint eigenfunctions and spectral parameters on quotients of G/K

Suppose $\Gamma < G$ is a cocompact, torsionfree lattice. Then,

$$\Gamma\backslash G/K = \begin{cases} \text{locally symmetric space (e.g. hyperbolic surface)} \\ \text{finite simplicial complex (e.g. finite regular graph),} \end{cases}$$

with universal cover equal to G/K.

Let $\mathcal{C}$ be either $D(G, K)$ or $H(G, K)$. Then $\mathcal{C}$ is generated by k commuting normal operators $A_1, \ldots, A_k$. Thus $L^2(\Gamma\backslash G/K)$ has an ONB of **joint eigenfunctions**:

$$\mathcal{C} \curvearrowright L^2(\Gamma\backslash G/K) = \bigoplus_j \mathbb{C}\psi_j.$$

By recording the eigenvalue of ψ_j for each A_k as a k-tuple, we obtain the **spectral parameter** ν_j.

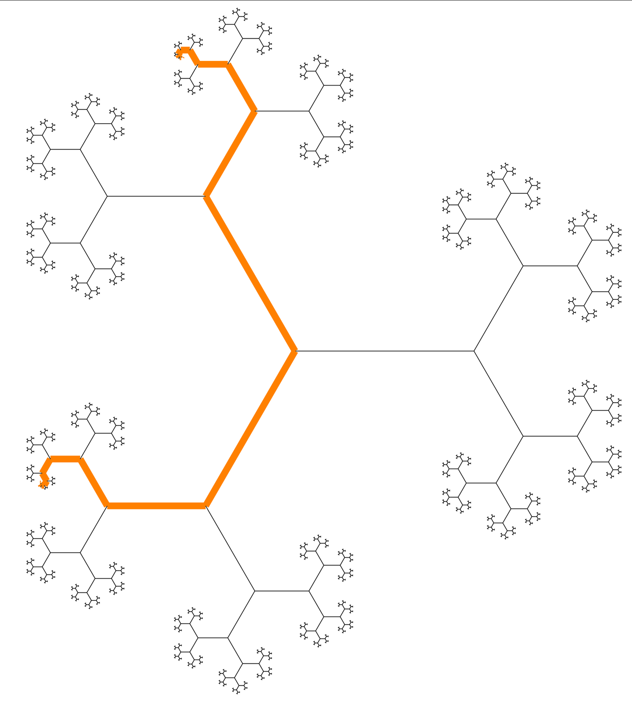
In the non-archimedean case, sometimes Γ preserves a non-trivial coloring with r colors, in which case we obtain r **coloring eigenfunctions**. For example, in the rank one case $\Gamma\backslash G/K$ may be a $(q+1)$-regular bipartite graph in which case we have $\pm(q+1)$ as eigenvalues. The eigenfunction associated to $-(q+1)$ is a non-trivial coloring eigenfunction.

Main result

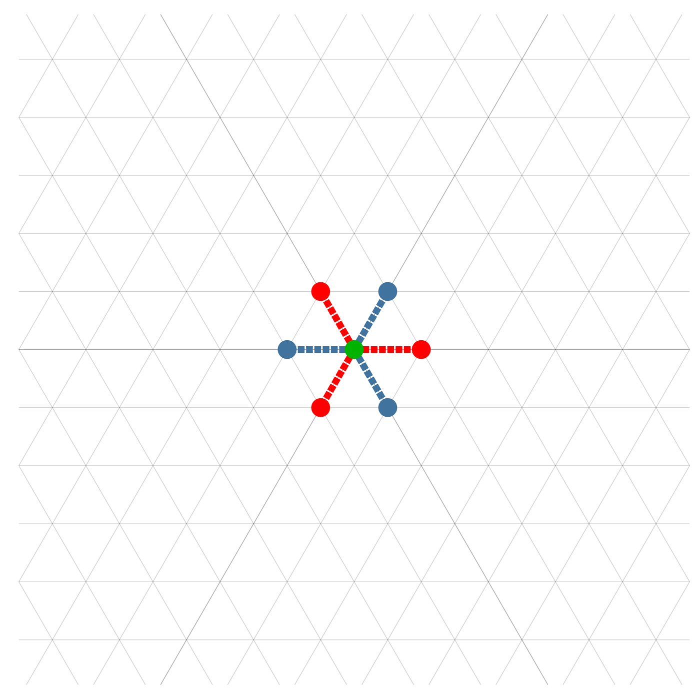
Theorem (P. '23) Let $G = \mathrm{PGL}(3, F)$ and $K = \mathrm{PGL}(3, \mathcal{O})$, where F is a non-archimedean local field of characteristic zero, and $\mathcal{O}$ is its ring of integers. Let $\Gamma_n < G$ be a sequence of torsionfree lattices, and let $Y_n = \Gamma_n\backslash G/K$. Suppose $\mathrm{card}(Y_n) \to \infty$. Let $\Theta \subset \Omega$ be a compact subset with positive Plancherel measure and not meeting a certain codimension one sublocus Ξ. Let $\{\psi_j^{(n)}\}$ be an ONB of eigenfunctions of $H(G, K)$ acting on $L^2(Y_n)$. Let $a_n \in L^\infty(Y_n)$ with universal L^∞-norm bound and orthogonal to all non-trivial coloring eigenfunctions. Then

$$\lim_{n\to\infty} \frac{1}{\#\{j : \nu_j^{(n)} \in \Theta\}} \sum_{\psi_j^{(n)} : \nu_j^{(n)} \in \Theta} \left| \sum_{v \in Y_n} a_n(v) \cdot |\psi_j^{(n)}(v)|^2 - \frac{1}{\mathrm{card}(Y_n)} \sum_{v \in Y_n} a_n(v) \right|^2 = 0.$$

Geometry of Bruhat-Tits buildings



Apartments in the tree are bi-infinite geodesics

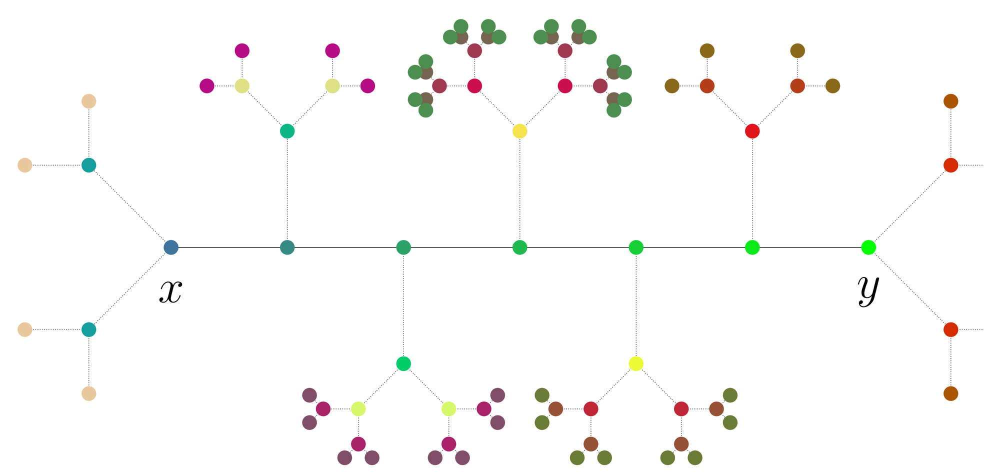


An apartment in the building for $\mathrm{PGL}(3, F)$

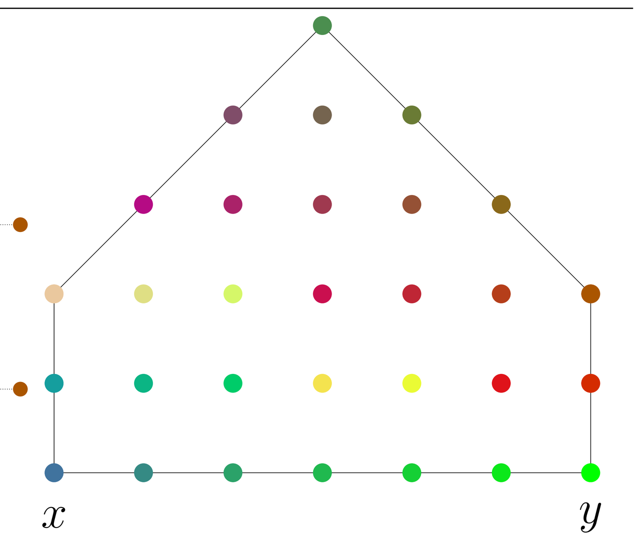
Some remarks about the proof

- Anantharaman-Le Masson '15 introduced QE in the BS limit (for regular graphs). Their proof involved "microlocal analysis on regular trees". Brooks-Le Masson-Lindenstrauss '16 found a new proof using "**wave propagation**" on regular graphs.
- The wave propagator roughly corresponds to averaging over "balls" of different radii. It has been adapted for hyperbolic surfaces by Le Masson-Sahlsten '17, rank one locally symmetric spaces by Abert-Bergeron-Le Masson '18, and locally symmetric spaces associated to $\mathrm{SL}(d, \mathbb{R})$ by Brumley-Matz '21.
- In rank one metric balls suffice; in higher rank one must use "**polytopal balls**".
- The wave propagator has desirable spectral properties, ultimately allowing one to reduce to bounding the norm of the kernel function of the wave propagator.
- BS convergence allows one to lift to analyzing the kernel function on the universal cover G/K.
- After changing variables, one may bound the norm of the kernel function using an **ergodic theorem** from Nevo '98, which bounds the op. norm of convolution ops associated to ergodic actions of semisimple algebraic groups over local fields.
- An input to the Nevo ergodic theorem is the volume of the set defining the convolution. One is thus led to bounding the volume of the intersection of "polytopal balls" in the building (the "geometric bound"). This is the hardest step and requires first classifying certain configurations in the building.
- We repeatedly use **Brion's formula**, which expresses the sum of an exponential function over lattice points in a polytope in terms of the geometry of the polytope.

Method for the geometric bound illustrated on the tree



Intersection of $B(x, 8)$ and $B(y, 8)$ in the 3-regular tree with $d(x, y) = 6$



Points in the intersection may be "classified" using lattice points in a polytope