A degenerate version of Brion's formula

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IMJ-PRG

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One-dimensional warm-up

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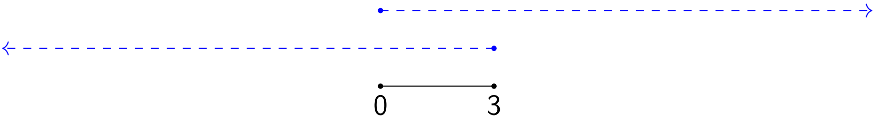

One-dimensional warm-up, cont.

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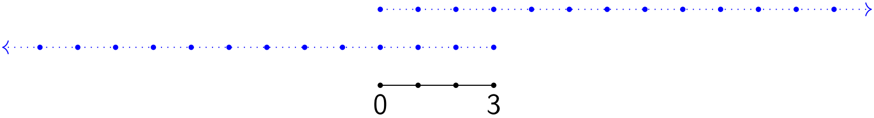
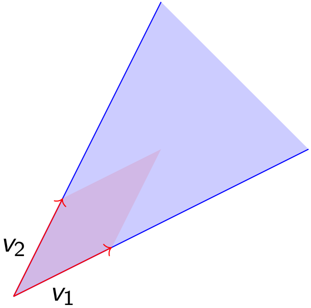


Exponential integral over a cone

- Given *simplicial* cone $\mathfrak{k} \subset V \simeq \mathbb{R}^n$ with conical basis $v_1, \ldots, v_n$ and $\xi \in \mathrm{Int}(\mathfrak{k}^\circ)$ we have absolute convergence:

$$\int_{\mathfrak{k}} e^{\langle \xi, x \rangle} dx = \mathrm{vol}(\square(v_1, \ldots, v_n)) \prod_j \frac{1}{-\langle \xi, v_j \rangle}.$$



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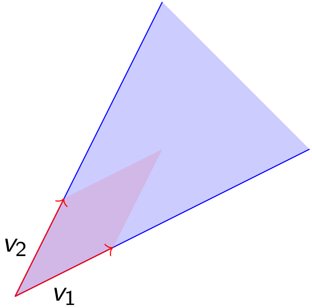
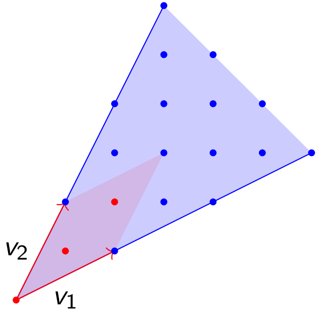


- This extends to a meromorphic function on $V_{\mathbb{C}}^*$ with singularities on $\langle \xi, v_j \rangle = 0$.

Exponential sum over a cone

- Given *rational* simplicial cone $\mathfrak{k}$ with *primitive* conical basis $v_1, \ldots, v_n$ and $\xi \in \operatorname{Int}(\mathfrak{k}^\circ)$ we have absolute convergence:

$$\sum_{\mathfrak{k}\cap\mathbb{Z}^n} e^{\langle \xi, \lambda \rangle} = \Big(\sum_{\lambda \in \square(v_1,\ldots,v_n)\cap\mathbb{Z}^n} e^{\langle \xi, \lambda \rangle} \Big) \prod_j \frac{1}{1 - e^{\langle \xi, v_j \rangle}}.$$



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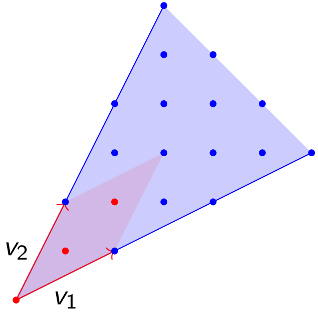

- This extends to a meromorphic function with singularities on $\langle \xi, v_j\rangle = 2\pi i k$ for some $k\in\mathbb{Z}$, i.e. $e^{\langle \xi,\cdot\rangle} = 1$ on $\mathbb{Z}^n \cap \mathbb{R}v_j$.

Exponential sum/integral over polyhedra

- Given a polyhedron $\mathfrak{q}$ not containing a line, define

$$I(\mathfrak{q};\xi) := \int_{\mathfrak{q}} e^{\langle \xi, x \rangle} dx$$

and extend meromorphically from the locus where it converges absolutely.

Exponential sum/integral over polyhedra

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and extend meromorphically from the locus where it converges absolutely.

- Given a lattice Λ and a *rational* polyhedron $\mathfrak{q}$ not containing a line, define

$$S_\Lambda(\mathfrak{q};\xi) := \sum_{\lambda \in \mathfrak{q}\cap\Lambda} e^{\langle \xi, \lambda\rangle}$$

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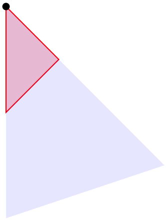
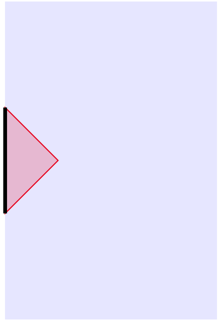
Tangent cones

- Each face $\mathfrak{f}$ of a polytope $\mathfrak{p}$ has a *tangent cone* $\mathfrak{s}_{\mathfrak{f}}^{\mathfrak{p}}$.





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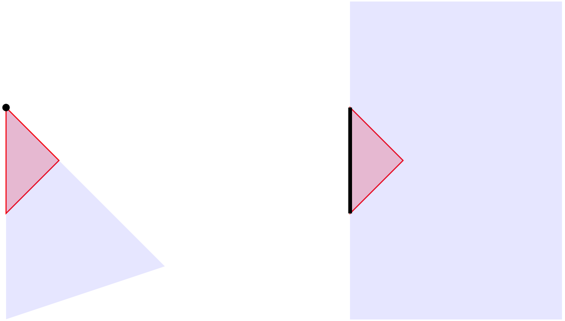

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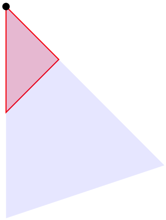

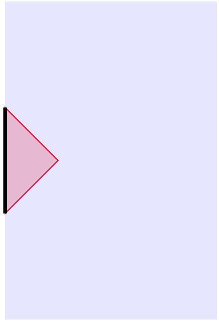

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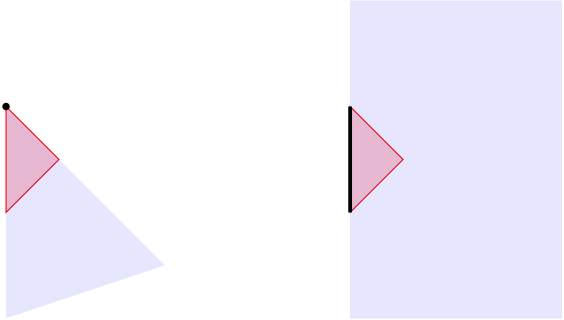

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- $S_\Lambda(\mathfrak{s}_{\mathfrak{v}}^{\mathfrak{p}};\xi)$ is singular when $e^{\langle \xi, \cdot \rangle} = 1$ on $\Lambda \cap \mathrm{lin}(\mathfrak{e})$ for some edge $\mathfrak{e}$ containing $\mathfrak{v}$.

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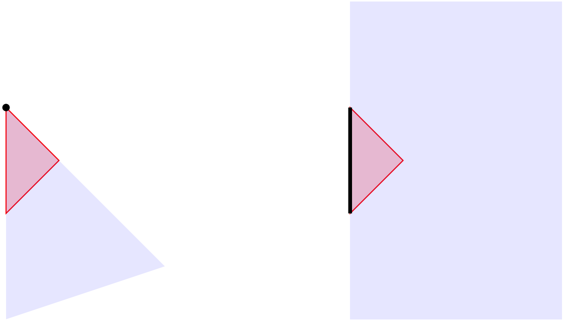

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- $\mathrm{lin}(\mathfrak{f})$ is the *linear* subspace parallel to $\mathfrak{f}$.

Brion's formula

Theorem (Brion, '88)

Suppose $\mathfrak{p} \subset V$ is a polytope. Then we have the following equality of meromorphic functions in $\xi \in V_{\mathbb{C}}^*$:

$$I(\mathfrak{p};\xi) = \sum_{\mathfrak{v}\in \mathrm{Vert}(\mathfrak{p})} I(\mathfrak{s}_{\mathfrak{v}}^{\mathfrak{p}};\xi).$$

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Theorem (Brion, '88)

Suppose $\mathfrak{p} \subset V$ is a rational polytope with respect to a lattice Λ. Then we have the following equality of meromorphic functions in $\xi \in V_{\mathbb{C}}^*$:

$$S_{\Lambda}(\mathfrak{p}; \xi) = \sum_{\mathfrak{v} \in \mathrm{Vert}(\mathfrak{p})} S_{\Lambda}(\mathfrak{s}_{\mathfrak{v}}^{\mathfrak{p}}; \xi).$$

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  1. The Brianchon-Gram formula.
  2. The extension of I and S_Λ to valuations.

Virtual polyhedra and valuations

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- A linear map on this vector space is called a *valuation*.

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Suppose $\mathfrak{p}$ is a polytope. Then

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Every term on the RHS is a polyhedron containing a line except for the tangent cones of the vertices.

Extensions of I and S_Λ to valuations

Theorem (Lawrence, Brion '91)

Define $I(\mathfrak{q};\xi) = 0$ if $\mathfrak{q}$ contains a line. Then $[\mathfrak{q}] \mapsto I(\mathfrak{q};\xi)$ defines a valuation.

Define $S_\Lambda(\mathfrak{q};\xi) = 0$ if $\mathfrak{q}$ is rational and contains a line. Then $[\mathfrak{q}] \mapsto S_\Lambda(\mathfrak{q};\xi)$ defines a valuation (on rational polyhedra).

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Applying I and S_Λ to Brianchon-Gram formula:

$$I(\mathfrak{p};\xi)=\sum_{\mathfrak{f}}(-1)^{\dim(\mathfrak{f})}I(\mathfrak{s}^{\mathfrak{p}}_{\mathfrak{f}};\xi)=\sum_{\mathfrak{v}}I(\mathfrak{s}^{\mathfrak{p}}_{\mathfrak{v}};\xi).$$

$$S_\Lambda(\mathfrak{p};\xi)=\sum_{\mathfrak{f}}(-1)^{\dim(\mathfrak{f})}S_\Lambda(\mathfrak{s}^{\mathfrak{p}}_{\mathfrak{f}};\xi)=\sum_{\mathfrak{v}}S_\Lambda(\mathfrak{s}^{\mathfrak{p}}_{\mathfrak{v}};\xi).$$

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- Given a degenerate ξ, we want a formula like Brion's formula such that
  1. **Each term only depends on some local geometry of $\mathfrak{p}$.**
  2. **Each term is actually holomorphic at ξ, i.e. we can actually "plug in" at ξ.**

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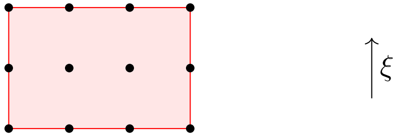

- If $\Lambda = \Lambda_1 \oplus \Lambda_2$ in a compatible way, and $\mathfrak{p}_i$ are lattice polytopes, then

$$S_\Lambda(\mathfrak{p}; \xi) = \#\{\mathfrak{p}_1 \cap \Lambda_1\} \cdot S_{\Lambda_2}(\mathfrak{p}_2; \xi|_{\mathfrak{p}_2}).$$

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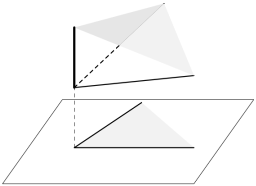

Figure: Image from Berline-Vergne '07

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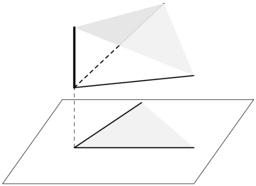

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- $\mathfrak{t}_{\mathfrak{f}}^{\mathfrak{p}}$ is a *pointed* cone in $\mathrm{lin}(\mathfrak{f})^{\perp}$.

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$$\mathsf{LC}_{\mathfrak{f}}^{\mathfrak{p}}(\xi) := \sum_{\mathfrak{f} \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_\ell \in \mathsf{mFl}_{\mathfrak{f}}^{\mathfrak{p}}(\xi)} (-1)^\ell \ \mathfrak{t}_{\mathfrak{f}}^{\mathfrak{h}_1} \times \mathfrak{t}_{\mathfrak{h}_1}^{\mathfrak{h}_2} \times \cdots \times \mathfrak{t}_{\mathfrak{h}_{\ell-1}}^{\mathfrak{h}_\ell} \times \mathfrak{t}_{\mathfrak{h}_\ell}^{\mathfrak{p}},$$

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  where $\mathsf{mFl}_{\mathfrak{f}}^{\mathfrak{p}}(\xi)$ is all *flags* of faces in $\{\mathfrak{p}\}_\xi$ starting from $\mathfrak{f}$.
- $\mathsf{LC}_{\mathfrak{f}}^{\mathfrak{p}}(\xi)$ is a *virtual cone* in $\mathrm{lin}(\mathfrak{f})^\perp$.

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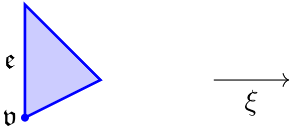
A more complicated example



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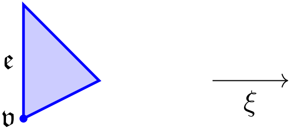

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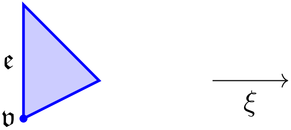

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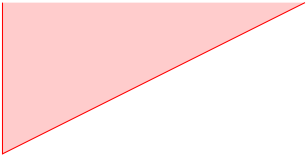

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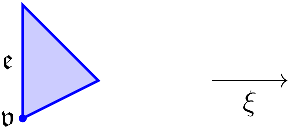

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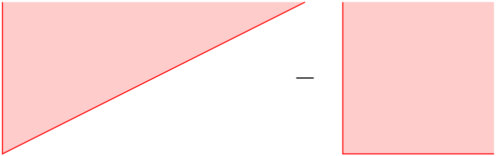

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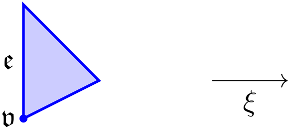

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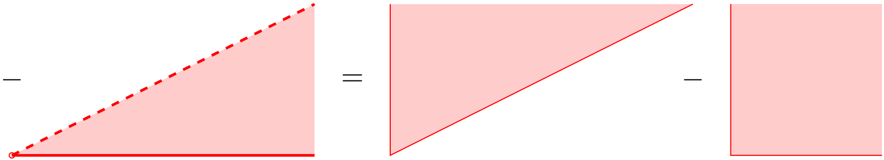

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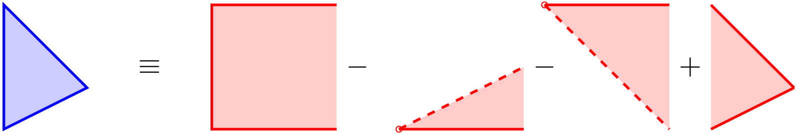

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Corollary (P. '24)

We have

$$I(\mathfrak{p}; \xi) = \sum_{\mathfrak{f} \in \{\mathfrak{p}\}_\xi} \mathrm{vol}(\mathfrak{f}) \cdot I(\mathrm{LC}_{\mathfrak{f}}^{\mathfrak{p}}(\xi); \xi),$$

and each term on the RHS is well-defined (non-singular).

The discrete setting?

Corollary (P. '24)

Suppose $\mathfrak{p}$ is a rational polytope. Then for any $\xi \in V_{\mathbb{C}}^*$ we have the following equality of meromorphic functions in α:

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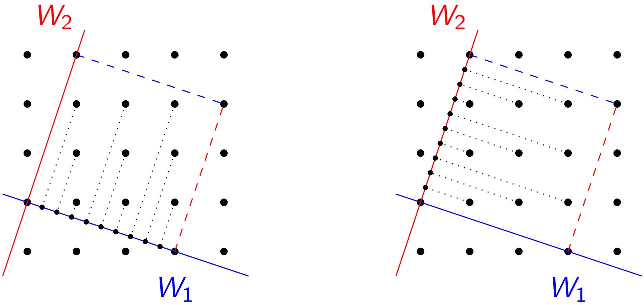

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If $\mathfrak{q}_i \subset W_i$ are polyhedra and $\xi = \xi_1 + \xi_2$ then

$$S_\Lambda(\mathfrak{q}_1 \times \mathfrak{q}_2; \xi) = \sum_{[\gamma] \in \Lambda/(\Lambda_1 \oplus \Lambda_2)} S_{\phi_1([\gamma]) + \Lambda_1}(\mathfrak{q}_1; \xi_1) \cdot S_{\phi_2([\gamma]) + \Lambda_2}(\mathfrak{q}_2; \xi_2).$$

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Given $\xi \in V_{\mathbb{C}}^*$, let $\tilde{V}$ be the subspace spanned by sublattice on which $e^{\langle \xi, \lambda \rangle} = 1$.

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Upshot: we should use $\{\mathfrak{p}\}_{\xi,\Lambda}$ and we can reduce to the case that $\xi \in (V_\mathbb{C}^*)^\Lambda$.

Holomorphicity

Theorem (P. '24)

Suppose $\mathfrak{f} \in \{\mathfrak{p}\}_{\xi,\Lambda}$. Then $S_\Lambda(\mathsf{LC}_{\mathfrak{f}}^{\mathfrak{p}}(\tilde{\xi}); \alpha)$ is holomorphic at $\alpha = \xi$.

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Bernoulli numbers $\longrightarrow$ generating function $\dfrac{x}{1-e^{-x}}$ $\longrightarrow$ Todd operators

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- Given a face $\mathfrak{f}$, define

$$\Lambda_{\mathfrak{f}} := \Lambda \cap \mathrm{lin}(\mathfrak{f}) \qquad \Lambda_{\mathfrak{f}^\perp} := \Lambda \cap \mathrm{lin}(\mathfrak{f})^\perp.$$

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Suppose $\mathfrak{p}$ is a rational polytope with respect to Λ. Then

$$S_\Lambda(\mathfrak{p};\xi) = \sum_{\mathfrak{f} \in \{\mathfrak{p}\}_{\xi,\Lambda}} \sum_{[\gamma] \in \Lambda/(\Lambda_{\mathfrak{f}} \oplus \Lambda_{\mathfrak{f}^\perp})} \#\{\mathfrak{f}^{\mathfrak{f}} \cap \left([\gamma^{\mathfrak{f}}] + \Lambda_{\mathfrak{f}}\right)\} \cdot S_{[\gamma^{\mathfrak{f}^\perp}] + \Lambda_{\mathfrak{f}^\perp}}(\operatorname{LC}_{\mathfrak{f}}^{\mathfrak{p}}(\tilde{\xi});\xi),$$

and each term on the RHS is well-defined (non-singular).

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Proposition (P. '24)

The function $\mu_W^\Gamma(\mathfrak{k};\xi)$ is holomorphic at every point in $(V_\mathbb{C}^*)^\Gamma$.

Degenerate Brion's formula, discrete setting version 2

Theorem (P. '24)

Suppose $\mathfrak{p}$ is a rational polytope. Then

$$S_\Lambda(\mathfrak{p};\xi) =$$

$$\sum_{\mathfrak{g}\in\{\mathfrak{p}\}_{\xi,\Lambda}} \mathrm{vol}^{\Lambda_\mathfrak{g}}(\mathfrak{g})\Big(\sum_{\mathfrak{g}\subseteq\mathfrak{f}\in\{\mathfrak{p}\}_{\xi,\Lambda}} \big(\sum_{[\gamma]\in\Lambda/(\Lambda_\mathfrak{f}\oplus\Lambda_{\mathfrak{f}^\perp})} \mu^{([\gamma^\mathfrak{f}]+\Lambda_\mathfrak{f})^{\mathfrak{g}^\perp}}_{\mathrm{lin}(\mathfrak{f})\cap\mathrm{lin}(\mathfrak{g})^\perp}(\mathfrak{t}^\mathfrak{f}_\mathfrak{g};0)\cdot S_{[\gamma^{\mathfrak{f}^\perp}]+\Lambda_{\mathfrak{f}^\perp}}(\mathrm{LC}^\mathfrak{p}_\mathfrak{f}(\tilde{\xi});\xi)\big)\Big).$$

and each term on the RHS is well-defined.

Exponential integrals over families of polytopes

- Brion's formula tells us that if ξ is generic then

$$I(t\cdot\mathfrak{p};\xi)=\sum_{v\in\mathrm{Vert}(\mathfrak{p})} I(t\cdot\mathfrak{v}+{}^{0}\mathfrak{s}_{\mathfrak{v}}^{\mathfrak{p}};\xi)=\sum_{\mathfrak{v}\in\mathrm{Vert}(\mathfrak{p})} I({}^{0}\mathfrak{s}_{\mathfrak{v}}^{\mathfrak{p}};\xi)\cdot e^{t\langle\xi,\mathfrak{v}\rangle}.$$

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A view towards applications

- Exponential integrals over polyhedra show up in analysis on *symmetric spaces*.

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